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THEORETICAL WAVE DRAG OF SHROUDED AIRFOILS AND BODIES

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SUMMARY

Formulas for the wave drag of shrouded symmetrical airfoils and shrouded bodies of revolution of arbitrary shape are derived by means of linearized theory. In the case of the airfoils the shroud consists of flat plates, and for the bodies of revolution the shroud is a cylindrical shell. The results obtained hold for a Mach number range dependent on the geometry of the configuration. Expressions are also given for determining a class of body shapes for which the wave drag is theoretically zero.

INTRODUCTION

A body moving at supersonic speeds has a wave drag which can be calculated either from integrations based upon the pressure at the surface of the body or by means of a momentum balance over a control surface surrounding the body. The control-surface approach shows more clearly that the wave drag is related to the transport of momentum in the Mach waves created by the body. This approach also suggests the scheme of reducing or destroying the wave drag through the use of a shroud as first shown by Ferrari (ref. 1). With such a shroud the waves are caught and reflected to the body surfaces where they may be absorbed without further reflection. From the standpoint of the pressure exerted on the body itself, it follows that the reflected waves may strike the rear portion of the body in such a way as to provide a buoyancy to overcome the resistance of the body alone. The detrimental effect of the additional friction drag due to a shroud is not included in the present study.

The principal object of the present investigation is to derive formulas for the wave drag of shrouded symmetrical airfoils and shrouded bodies of revolution of arbitrary shape. The airfoil is shrouded by flat plates and the body of revolution is shrouded by a cylindrical shell. Although many configurations are possible, the analysis here considers the particular arrangement where the shroud extends at least far enough forward to catch the Mach wave emanating from the body nose, and far enough rearward to cast Mach waves on the base of the body. As a special application of the results obtained, a class of body shapes, similar to those given by Busemann (ref. 2) and Ferri (ref. 3), are found for which the wave drag is theoretically zero.

For either the case of the airfoil or the body of revolution, the analysis is based on the assumption of linearized theory and on operational methods using Laplace transforms. Ward (refs. 4 and 5) has shown in considerable detail how operational calculus can be employed to treat bodies of revolution and quasi-cylindrical tubes. A similar approach will be followed here, except that Heaviside notation is not used.

To the order of the analysis employed, discontinuities in the slope of the airfoil are admitted, while for the body of revolution the gradient of cross-sectional area and its derivative are assumed to undergo no abrupt changes. In actual practice, discontinuities producing fixed compression waves would certainly upset the accuracy of the results even more than for the airfoil or the body alone since the opposing surfaces offer the possibility of shock-wave and boundary-layer interaction.

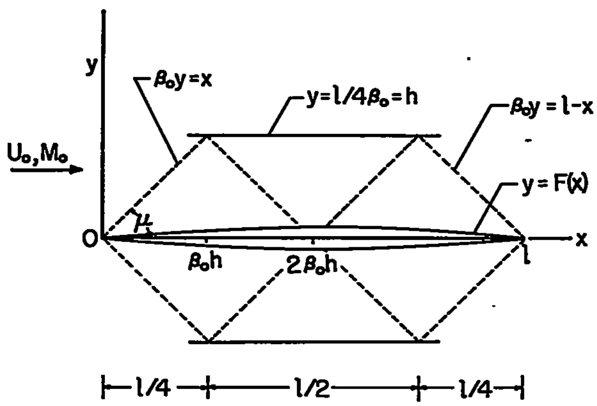
SYMBOLS

a_o	speed of sound in free stream
A,B	cross-sectional area of body in dimensionless terms, $\frac{S}{\beta_o h^2}$
c_d	section drag coefficient, $\frac{D}{q_o l}$
C_D	body drag coefficient, $\frac{D}{q_o S_f}$
C_p	pressure coefficient, $\frac{p - p_o}{q_o}$
D	wave drag
f	function defined in equation (5), $\frac{F}{\beta_o h}$
F	function defining upper surface of airfoil or generating curve of the body of revolution
g	function satisfying relation (53)
h	distance of shroud from x axis

$I_o, I_1, J_o, J_1, K_o, K_1$	Bessel functions (ref. 6)
l	length of body, or chord length of airfoil
m	$\frac{2\beta_o h}{l}$
M_o	Mach number in the free stream
p	local pressure
p_o	pressure in free stream
q_o	free-stream dynamic pressure, $\frac{\rho_o U_o^2}{2}$
R	radius of body
S(x)	cross-sectional area of body
S_f	frontal area of body
t_o	maximum thickness of body
U_o	free-stream velocity
x	Cartesian coordinate in free-stream direction
y	Cartesian coordinate, measuring vertical distance for airfoil and radial distance for body
Y_1	Bessel function (ref. 6)
β_o	$\sqrt{M_o^2 - 1}$
η	dimensionless variable introduced in equation (4), $\frac{y}{h}$
μ	Mach angle, arc sin $\frac{1}{M_o}$
ξ	dimensionless variable introduced in equation (4), $\frac{x}{\beta_o h}$
ρ_o	free-stream density
$\varphi(\xi,\eta)$	perturbation velocity potential in dimensionless terms, $\frac{\Phi(x,y)}{U_o h}$

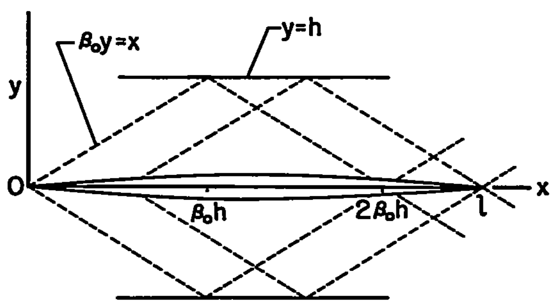
$\phi(x,y)$	perturbation velocity potential
Ω	influence function defined in equation (35)
$(\bar{\ })$	Laplace transform of function

MATHEMATICAL STATEMENT OF PROBLEM



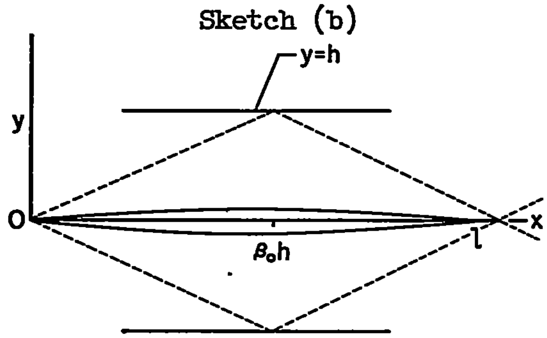
$(\beta_o h = l/4)$

Sketch (a)



$(l/4 < \beta_o h < l/2)$

Sketch (b)



$(\beta_o h > l/2)$

Sketch (c)

Consider a symmetrical airfoil or slender body of revolution placed at zero angle of attack at Mach number $M_o = U_o/a_o > 1$, where a_o is the speed of sound in the free stream and where U_o, the free-stream velocity, is alined with the x axis. The y axis measures vertical distance in the case of the airfoil and radial distance in the case of the body. The nose of the body is at the origin of the coordinate system.

In the configurations to be considered the airfoil is shrouded by two flat plates and the body of revolution is shrouded by a cylindrical tube. The shrouding plates (or tubes) are placed so that the distance h by which the shroud is removed from the x axis is such that $l/4 \leq \beta_o h \leq l/2$, where $\beta_o = \cot \mu = \sqrt{M_o^2 - 1}$ and l is the body length. The shroud is required to extend at least from $x = \beta_o h$ to $x = l - \beta_o h$, but it would produce no additional effects on the body if it were longer. Sketches (a), (b), and (c) show the geometry of the configuration with either the airfoil or body of revolution in three typical arrangements. If $\beta_o h = l/4$, all the Mach waves from the forward portion of the body are reflected onto the rearward, and when $\beta_o h \geq l/2$ there is no effect of the shroud since the waves are reflected behind the body.

The upper surface (or generating curve) of the airfoil or body is assumed given by the function

$$y = F(x) \tag{1}$$

It will also be assumed that the body closes at both ends (i.e., that $F(0) = F(l) = 0$) and that the thickness-length ratio of the body is sufficiently small relative to the Mach angle μ that linearized theory applies. As a result, the perturbation velocity potential $\Phi(x,y)$ satisfies the partial differential equation

$$\beta_o{}^2 \frac{\partial^2\Phi}{\partial x^2} - \frac{\partial^2\Phi}{\partial y^2} - \frac{\nu}{y}\frac{\partial\Phi}{\partial y} = 0 \tag{2}$$

together with the boundary conditions

$$\left.\begin{aligned} \left(\frac{y^\nu}{U_o}\frac{\partial\Phi}{\partial y}\right)_{y=\nu F} &= F^\nu \frac{dF}{dx}\,, \qquad 0 \le x \le l \\ \left(\frac{1}{U_o}\frac{\partial\Phi}{\partial y}\right)_{y=h} &= 0\,, \qquad \beta_o h \le x \le l - \beta_o h \end{aligned}\right\} \tag{3}$$

where the parameter ν equals 0 and 1, respectively, for the airfoil and body of revolution. One then has the problem of finding a solution of Φ, and from this to determine the drag of the configuration.

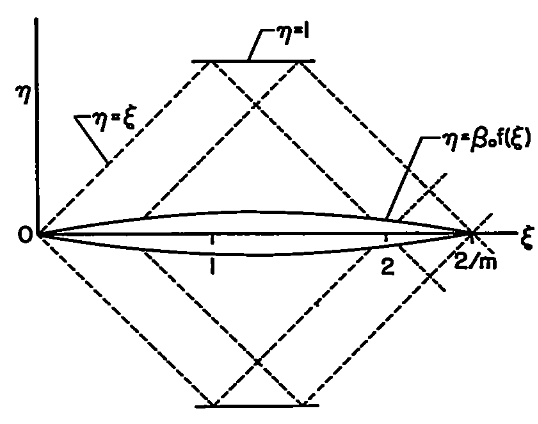
If dimensionless variables ξ, η, φ defined by the relations

$$\xi = x/\beta_o h\,, \qquad \eta = y/h\,, \qquad \varphi(\xi,\eta) = \Phi(x,y)/U_o h \tag{4}$$

are now introduced, the expression for the body surface, as given by equation (1), becomes

$$\eta = \frac{2\beta_o}{ml} F\left(\frac{ml}{2}\,\xi\right) = \beta_o f(\xi) \tag{5}$$

with $m = 2\beta_o h/l$, and sketch (b) becomes sketch (d). The differential equation is then



(1/2 < m < 1)

Sketch (d)

$$\frac{\partial^2\varphi}{\partial\xi^2} - \frac{\partial^2\varphi}{\partial\eta^2} - \frac{\nu}{\eta}\frac{\partial\varphi}{\partial\eta} = 0 \tag{6}$$

with boundary conditions

$$\left.\begin{aligned} \left(\eta^\nu \frac{\partial\varphi}{\partial\eta}\right)_{\eta=\nu\beta_o f} &= (\beta_o f)^\nu \frac{df}{d\xi}, & 0 \leq \xi \leq 2/m \\ \left(\frac{\partial\varphi}{\partial\eta}\right)_{\eta=1} &= 0, & 1 \leq \xi \leq 2/m - 1 \end{aligned}\right\} \tag{7}$$

Operational methods based upon the Laplace transform are suited for treating the basic differential equation (6) for either the airfoil or the body of revolution. Denoting the Laplace transform of a function $g(\xi,\eta)$ by $\bar{g}(s;\eta)$ where

$$\bar{g}(s;\eta) = \int_0^\infty e^{-s\xi} g(\xi,\eta) d\xi \tag{8}$$

and employing this transformation for relations (6) and (7), one obtains the differential equation

$$s^2\bar{\varphi} - \frac{\partial^2\bar{\varphi}}{\partial\eta^2} - \frac{\nu}{\eta}\frac{\partial\bar{\varphi}}{\partial\eta} = 0 \tag{9}$$

together with the boundary conditions

$$\left(\eta^\nu \frac{\partial\bar{\varphi}}{\partial\eta}\right)_{\eta=\nu\beta_o f} = (\beta_o)^\nu \overline{f^\nu \frac{df}{d\xi}} \tag{10a}$$

and

$$\left(\frac{\partial\bar{\varphi}}{\partial\eta}\right)_{\eta=1} = 0 \tag{10b}$$

Once $\bar{\varphi}$ has been determined from equation (9), the drag of the shrouded figures can be calculated. In order to carry out the calculations, it is convenient to treat the airfoil and body-of-revolution problems separately. The analysis for the airfoil offers little in the way of novelty but will be given first since it illustrates the framework of the methods employed.

AIRFOIL WITH SHROUDING PLATES

Evaluation of Wave Drag

The solution of equation (9) for the case of the airfoil (i.e., when $\nu = 0$) is

$$\bar{\varphi}(s;\eta) = a(s)e^{-s\eta} + b(s)e^{s\eta} \tag{11}$$

Since vertical symmetry exists in the flow field, attention can be limited to the upper half of the figure. Over the forward part of the airfoil the downstream inclination of the outgoing waves indicates, furthermore, that the second term in the answer (11) can be deleted. Imposing the boundary condition (10a), one then has

$$-sa(s)e^{-s\eta}\Big]_{\eta=0} = s\bar{f} \; , \qquad a(s) = -\bar{f}$$

and it follows that the perturbation velocity potential satisfies the relations

$$\left.\begin{aligned} \bar{\varphi}(s;\eta) &= -\bar{f}e^{-s\eta} \\ \varphi(\xi,\eta) &= -f(\xi - \eta) \end{aligned}\right\} \tag{12a}$$

or, in terms of the physical variables,

$$\Phi(x,y) = -\frac{U_0}{\beta_0} F(x - \beta_0 y) \; , \qquad 0 \leq x - \beta_0 y \leq ml - 2\beta_0 y \tag{12b}$$

where $m = 2\beta_0 h/l$, $1/2 \leq m \leq 1$.

The flow around the forward portion of the airfoil is given by equations (12); in order to predict the flow around the rearward portion, however, it is necessary to determine the nature of the incoming waves from the plate. The velocity potential of these incoming waves can be obtained from the second term in the right member of equation (11). Boundary condition (10b) requires that vertical velocity be zero at $\eta = 1$. From equations (12) and (11), therefore, one has

$$sb(s)e^{s\eta}\Big]_{\eta=1} = -s\bar{f}e^{-s\eta}\Big]_{\eta=1} \; ; \qquad b(s) = -\bar{f}e^{-2s}$$

Hence the potential φ_2 of the incoming waves is determined by

$$\left.\begin{aligned}\bar{\varphi}_2(s;\eta) &= -\bar{f}e^{-s(2-\eta)}\\ \varphi_2(\xi,\eta) &= -f(\xi+\eta-2)\end{aligned}\right\}\tag{13}$$

and the potential in the region of the plate is

$$\varphi = -f(\xi-\eta) - f(\xi+\eta-2)$$

If the potential φ over the rear of the airfoil is written in the form

$$\varphi(\xi,\eta) = \varphi_2(\xi,\eta) + \varphi_3(\xi,\eta)$$

and $\bar{\varphi}_3$ is assumed expressible by terms of the form $a_3(s)e^{-s\eta}$, the boundary condition (10a) yields

$$-s\bar{f}e^{-s(2-\eta)}\Big]_{\eta=0} - sa_3(s)e^{-s\eta}\Big]_{\eta=0} = s\bar{f}$$

Thus $a_3(s) = -\bar{f}e^{-2s} - \bar{f}$, so that

$$\varphi_3(\xi,\eta) = -f(\xi-\eta-2) - f(\xi-\eta)$$

The potential over the rear of the airfoil is then given as follows:

$$\varphi(\xi,\eta) = -f(\xi-\eta-2) - f(\xi+\eta-2) - f(\xi-\eta)\tag{14a}$$

or finally

$$\Phi(x,y) = -\frac{U_o}{\beta_o}\left[F(x-\beta_o y-ml) + F(x+\beta_o y-ml) + F(x-\beta_o y)\right], \quad ml \leq x-\beta_o y \leq l\tag{14b}$$

It remains now to find the expression for wave drag on the surface. In the physical variables, the total wave drag is

$$\frac{D}{q_o} = 2\int_0^l \left(\frac{p-p_o}{q_o}\right)_{y=0} \frac{dF}{dx}\,dx = 2\int_0^l C_p(x,o)\,\frac{dF}{dx}\,dx\tag{15}$$

where $q_o = \rho_o U_o^2/2$ is the dynamic pressure in the free stream of density ρ_o. With the substitution, from thin-airfoil theory, $C_p = -(2/U_o)(\partial\Phi/\partial x)$, the pressure coefficient on the airfoil surface is

$$\left.\begin{aligned} C_p &= \frac{2}{\beta_o} F'(x) \ , && 0 \leq x \leq ml \\ C_p &= \frac{2}{\beta_o} [F'(x) + 2F'(x - ml)] \ , && ml \leq x \leq l \\ & && (1/2 \leq m \leq 1) \end{aligned}\right\}$$

and the drag, expressed in coefficient form, can be written as

$$\left.\begin{aligned} c_d &= \frac{4}{\beta_o l}\int_{ml}^{l} [F'(x) + F'(x - ml)]^2 dx + \frac{4}{\beta_o l}\int_{(1-m)l}^{l} [F'(x)]^2 dx \ , && 1/2 \leq m \leq 1 \\ c_d &= \frac{4}{\beta_o l}\int_0^{l} [F'(x)]^2 dx \ , && m > 1 \end{aligned}\right\} \tag{16}$$

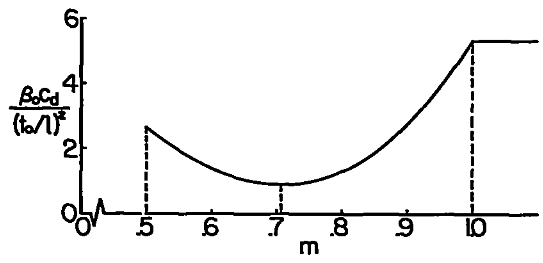
If the reflecting plates of the configuration are designed for a fixed height h, formulas (16) give the drag coefficient of a shrouded airfoil of arbitrary shape for Mach numbers $M_o \geq \sqrt{1 + l^2/16h^2}$. When $\beta_o h = l/4$ $(m = 1/2)$, the waves are reflected from the forward portion of the airfoil onto the rearward as in sketch (a), and for $m > 1$ the reflected waves do not intersect the airfoil. Formulas (16) thus yield results for an airfoil with or without shroud.

Consider, as a simple example, the shrouded biconvex airfoil section whose upper surface is defined by

$$F(x) = \frac{2t_o}{l^2} x(l - x) \ , \qquad 0 \leq x \leq l \tag{17}$$

where t_o is the maximum thickness of the airfoil. The drag coefficient, given by formulas (16), is then

$$\left.\begin{aligned} c_d &= \frac{16}{3\beta_o}\left(\frac{t_o}{l}\right)^2(4m^3 - 6m + 3), \qquad 1/2 \leq m \leq 1 \\ c_d &= \frac{16}{3\beta_o}\left(\frac{t_o}{l}\right)^2, \qquad m > 1 \end{aligned}\right\} \tag{18}$$



Sketch (e)

with the parameter $m = 2\beta_o h/l$. It is seen in sketch (e) that for this special case, the value of $\beta_o c_d$ for $1/2 \leq m \leq 1$ is always less than its value for $m \geq 1$, and that a relative minimum occurs at $m = 1/\sqrt{2}$. If the ratio h/l is fixed, formulas (18) give the drag of the shrouded airfoil as a function of β_o for $\beta_o \geq l/4h$.

Shrouded Airfoils Having Zero Wave Drag

Determination of shape.- Although formula (16) for the drag coefficient will in general be greater than zero, there exist classes of airfoil shapes for which the drag is theoretically zero. Since the two integrals in the first expression in equations (16) can never be negative, the necessary and sufficient condition that the drag vanish is that

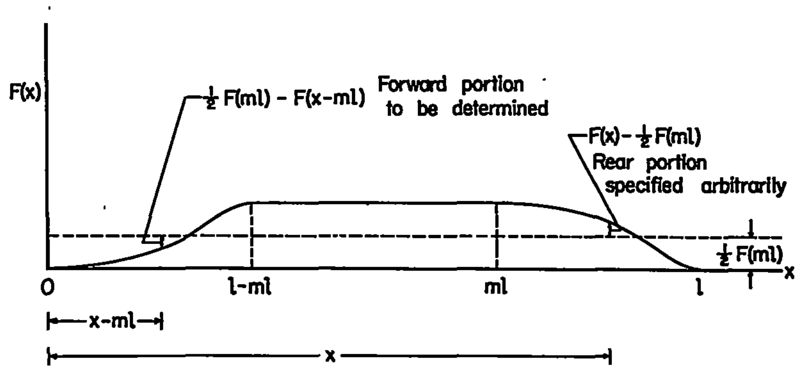
$$\left.\begin{aligned} F'(x) &= -F'(x - ml), \qquad ml \leq x \leq l \\ F'(x) &= 0, \qquad (1 - m)l \leq x \leq ml \end{aligned}\right\} \tag{19a}$$

After integration, these relations become

$$\left.\begin{aligned} \frac{1}{2}F(ml) - F(x - ml) &= F(x) - \frac{1}{2}F(ml), \qquad ml \leq x \leq l \\ F(x) &= F(ml), \qquad (1 - m)l \leq x \leq ml \end{aligned}\right\} \tag{19b}$$

Thus the airfoil can be drawn in an arbitrary manner from $x = ml$ to $x = l$ and the forward portion of the airfoil shape in the interval $0 \leq x \leq (1 - m)l$ is determined; the portion in the interval

$(1 - m)l \leq x \leq ml$ is flat and equal to $F(ml)$. Sketch (f) shows the geometrical construction of the profile. The upper half of such an airfoil is equivalent to the lower wing of the linearized version of a Busemann biplane arrangement, while the lower half is the upper wing.



Sketch (f)

For the special case when $m = 1/2$, as in sketch (a), equations (19b) can be written in the form

$$F(x) - \frac{1}{2}F\left(\frac{l}{2}\right) = \frac{1}{2}F\left(\frac{l}{2}\right) - F\left(x - \frac{l}{2}\right), \qquad l/2 \leq x \leq l \tag{19c}$$

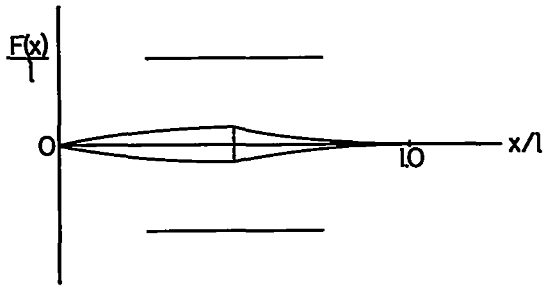
If, moreover, the airfoil is assumed to be symmetrical fore and aft, $F(x) = F(l - x)$ and equation (19c) becomes

$$F(x) - \frac{1}{2}F\left(\frac{l}{2}\right) = \frac{1}{2}F\left(\frac{l}{2}\right) - F\left(\frac{l}{2} - x\right), \qquad 0 \leq x \leq l/2 \tag{19d}$$

In this event the forward half of the airfoil has odd symmetry about the ordinate of the quarter-chord position and, similarly, the rear half has odd symmetry about the three-quarter chord position. It is also found that the pressure distribution on the airfoil has fore-and-aft symmetry.

Drag for off-design condition.- A shrouded airfoil whose geometry satisfies relation (19a), however, will only have zero drag for some particular value of the parameter m, say m_o. If such an airfoil is moving so that the parameter m is different from m_o, formulas (16) for the drag coefficient become

$$\left.\begin{aligned}
c_d &= \frac{4}{\beta_o l}\int_0^l [F'(x)]^2 dx - \frac{8}{\beta_o l}\int_{m_o l}^{l} F'(x - m_o l)F'(x - ml)dx, & 1/2 \leq m \leq m_o \leq 1 \\
c_d &= \frac{4}{\beta_o l}\int_0^l [F'(x)]^2 dx - \frac{8}{\beta_o l}\int_{ml}^{l} F'(x - m_o l)F'(x - ml)dx, & 1/2 \leq m_o \leq m \leq 1 \\
c_d &= \frac{4}{\beta_o l}\int_0^l [F'(x)]^2 dx, & m > 1
\end{aligned}\right\} \tag{20}$$



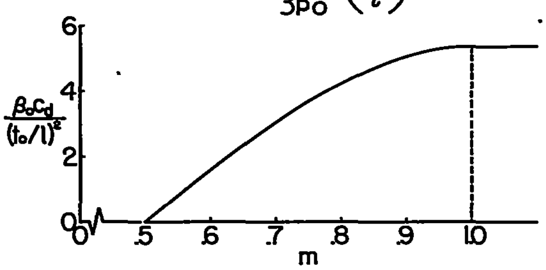
Sketch (g)

As an example, let the upper surface of the airfoil (shown in sketch (g)) be defined by

$$\left.\begin{aligned} F(x) &= \frac{2t_o}{l^2}\, x(l - x) \ , \qquad 0 \leq x \leq l/2 \\ &= \frac{2t_o}{l^2}\,(l - x)^2 , \qquad l/2 \leq x \leq l \end{aligned}\right\} \tag{21}$$

If the design of the configuration is such that $2\beta_o h = l/2$ (i.e., $m_o = 1/2$), relation (19a) is satisfied so that there is no drag. When $l/2 \leq 2\beta_o h \leq l$, however, the drag coefficient for the airfoil is given, from equation (20), by

$$\left.\begin{aligned} c_d &= \frac{16}{3\beta_o}\left(\frac{t_o}{l}\right)^2(-4m^3 + 6m^2 - 1) \ , \qquad 1/2 \leq m \leq 1 \\ c_d &= \frac{16}{3\beta_o}\left(\frac{t_o}{l}\right)^2 , \qquad m > 1 \end{aligned}\right\} \tag{22}$$



Sketch (h)

A plot of $\beta_o c_d/(t_o/l)^2$ against m is shown in sketch (h) for values of m greater than 1/2. Below $m = 1/2$ the calculations become more involved but the same general method is applicable.

BODY OF REVOLUTION WITH SHROUDING TUBE

Evaluation of Wave Drag

The solution of equation (9) for the case of the body of revolution (i.e., when $\nu = 1$) is

$$\bar{\varphi}(s;\eta) = a(s)K_o(s\eta) + b(s)I_o(s\eta) \tag{23}$$

where Watson's notation (ref. 6) for the Bessel functions K_o and I_o is used. Over the forward portion of the body the wave system is outgoing and the solution can be formulated from the first term in the right member of equation (23). In order to impose the boundary condition (10a), one employs the relation

$$\frac{\partial\bar{\varphi}}{\partial\eta} = -\ sa(s)K_1(s\eta) \simeq -\ \frac{a(s)}{\eta}$$

the last expression holding by virtue of the fact that η is small on the body surface. The boundary condition then yields, for $f(0) = 0$,

$$a(s) = -\ \frac{s\bar{A}(s)}{2\pi} \tag{24}$$

where the quantity A is

$$A(\xi) = \beta_o \pi f^2(\xi) = \frac{\pi F^2(x)}{\beta_o h^2} = \frac{\pi R^2(x)}{\beta_o h^2} = \frac{S(x)}{\beta_o h^2} \tag{25}$$

$S(x)$ being the cross-sectional area of the body. The solution over the forward portion of the body is thus

$$\bar{\varphi}(s;\eta) = -\ \frac{1}{2\pi}\ s\bar{A}(s)K_o(s\eta) \tag{26}$$

From equation (26) and the boundary condition (10b), the potential φ_2 of the incoming waves from the shrouding tube can be calculated. If the Laplace transform of φ_2 is assumed expressible in the form

$$\bar{\varphi}_2(s;\eta) = b(s)I_0(s\eta)\ , \qquad \frac{\partial\bar{\varphi}_2}{\partial\eta} = sb(s)I_1(s\eta)$$

the boundary condition at $\eta = 1$ yields

$$sb(s)I_1(s) = -\ \frac{s^2\bar{A}(s)K_1(s)}{2\pi}\ ; \qquad b(s) = -\ \frac{s\bar{A}(s)K_1(s)}{2\pi I_1(s)}$$

Thus $\bar{\varphi}_2$ is given as follows:

$$\bar{\varphi}_2(s;\eta) = -\ \frac{s\bar{A}(s)K_1(s)I_0(s\eta)}{2\pi I_1(s)}\ ; \qquad \frac{\partial\bar{\varphi}_2}{\partial\eta} = -\ \frac{s^2\bar{A}(s)K_1(s)I_1(s\eta)}{2\pi I_1(s)} \tag{27}$$

Let the potential φ over the rear of the body be written as $\varphi(\xi,\eta) = \varphi_2(\xi,\eta) + \varphi_3(\xi,\eta)$ where $\bar{\varphi}_3$ is assumed to be given by the terms of the form $a_3(s)K_0(s\eta)$. In the previous case of the airfoil, the normal gradient of φ_2 at the airfoil surface was of the same order as the imposed boundary condition and it was necessary to take proper regard of φ_2 when equation (10a) was satisfied. In the present case the normal gradient of φ_2 is of higher order (at the body surface) in comparison with the contribution of φ_3, and the boundary condition is satisfied within the accuracy of the theory by the relation

$$a_3(s) = -\frac{1}{2\pi} s\bar{A}(s) \quad , \qquad A(\xi) = \pi\beta_0 f^2(\xi) \tag{28}$$

The solution over the rear of the body is therefore

$$\bar{\varphi} = -\frac{s\bar{A}(s)K_1(s)I_0(s\eta)}{2\pi I_1(s)} - \frac{s\bar{A}(s)K_0(s\eta)}{2\pi} \tag{29}$$

In terms of the physical variables, the drag D of the configuration can be written as an integral of pressure over the surface of the body:

$$\frac{D}{q_0} = \int_0^l \left(\frac{p - p_0}{q_0}\right)_b 2\pi F(x) \frac{dF}{dx}\, dx \tag{30}$$

Since for slender bodies of revolution the pressure-velocity relation becomes

$$\frac{p - p_0}{q_0} = -\frac{2}{U_0}\frac{\partial\Phi}{\partial x} - \frac{1}{U_0^2}\left(\frac{\partial\Phi}{\partial y}\right)^2$$

it follows that the drag integral is, in terms of the dimensionless variables,

$$\frac{D}{q_0} = \beta_0 h^2 \int_0^{2/m} \left[-\frac{2}{\beta_0}\frac{\partial\varphi}{\partial\xi} - \left(\frac{\partial\varphi}{\partial\eta}\right)^2\right] \frac{dA}{d\xi}\, d\xi \tag{31}$$

where the Laplace transforms of φ are given by equation (26) for $0 \leq \xi \leq 2$ and by equation (29) for $2 \leq \xi \leq 2/m \leq 4$. Proceeding to the evaluation of the drag integral, we find from equation (26) that for small values of η

$$\left.\begin{aligned}\frac{\partial\bar{\varphi}}{\partial\xi} &= -\frac{1}{2\pi}\,s^2\bar{A}(s)K_0(s\eta) \approx \frac{s^2\bar{A}(s)}{2\pi}\left(\ln\frac{s\eta}{2} + \gamma_0\right)\\ \frac{\partial\bar{\varphi}}{\partial\eta} &= \frac{1}{2\pi}\,s^2\bar{A}(s)K_1(s\eta) \approx \frac{s\bar{A}(s)}{2\pi\eta}\end{aligned}\right\} \tag{32a}$$

where γ_0 is Euler's constant. From equation (29), similarly,

$$\left.\begin{aligned}\frac{\partial\bar{\varphi}}{\partial\xi} &= -\frac{s^2}{2\pi}\,\bar{A}(s)K_0(s\eta) - \frac{s^2}{2\pi}\,\bar{A}(s)\,\frac{K_1(s)}{I_1(s)}\,I_0(s\eta)\\ &\approx \frac{s^2}{2\pi}\,\bar{A}(s)\left(\ln\frac{s\eta}{2} + \gamma_0\right) - \frac{s^2}{2\pi}\,\bar{A}(s)\,\frac{K_1(s)}{I_1(s)}\\ \frac{\partial\bar{\varphi}}{\partial\eta} &= \frac{s^2}{2\pi}\,\bar{A}(s)K_1(s\eta) - \frac{s^2}{2\pi}\,\bar{A}(s)\,\frac{K_1(s)}{I_1(s)}\,I_0'(s\eta)\\ &\approx \frac{s\bar{A}(s)}{2\pi\eta}\end{aligned}\right\} \tag{32b}$$

From these expressions, the inverse transform can be written explicitly on the surface of the body in the following form:

for $0 \leq \xi \leq 2$

$$\left.\begin{aligned}\frac{\partial\varphi}{\partial\xi} &= \frac{1}{2\pi}\,A''(\xi)\ln\frac{\beta_0 f}{2} - \frac{1}{2\pi}\,\frac{\partial}{\partial\xi}\int_0^{\xi} A''(\xi_1)\ln(\xi - \xi_1)d\xi_1\\ \frac{\partial\varphi}{\partial\eta} &= \frac{1}{2\pi\beta_0 f(\xi)}\,\frac{dA}{d\xi} = \frac{df}{d\xi}\end{aligned}\right\} \tag{33a}$$

for $2 \leq \xi \leq 2/m \leq 4$

$$\left.\begin{aligned}\frac{\partial\varphi}{\partial\xi} &= \frac{1}{2\pi}\,A''(\xi)\ln\frac{\beta_0 f}{2} - \frac{1}{2\pi}\,\frac{\partial}{\partial\xi}\int_0^{\xi} A''(\xi_1)\ln(\xi - \xi_1)d\xi_1 - \frac{1}{2}\,A''(\xi - 2) - {}\\ &\quad \frac{1}{2}\int_2^{\xi} A'''(\xi_1 - 2)\Omega(\xi - \xi_1)d\xi_1 - \frac{1}{2}\,A''(0)\Omega(\xi - 2)\\ \frac{\partial\varphi}{\partial\eta} &= \frac{df}{d\xi}\end{aligned}\right\} \tag{33b}$$

where $\Omega(\xi)$ is an influence function whose Laplace transform is

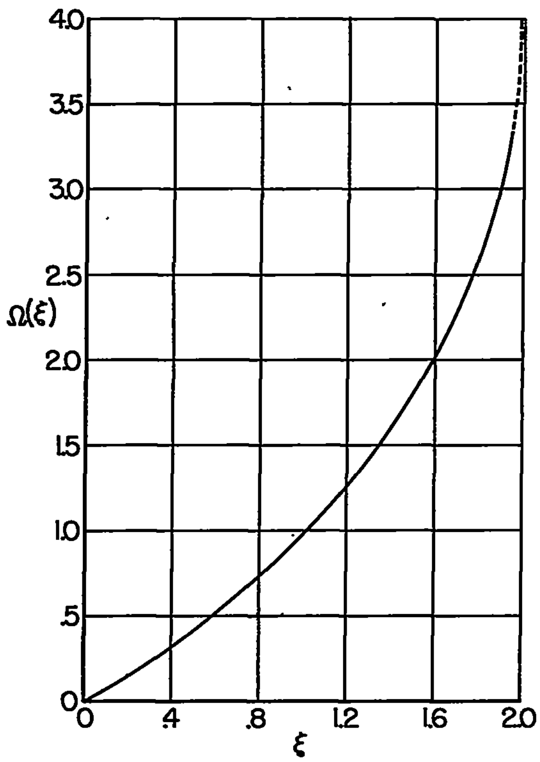
$$\bar{\Omega}(s) = \frac{1}{s}\left[\frac{e^{s}K_1(s)}{\pi e^{-s}I_1(s)} - 1\right] \tag{34}$$

and where the usual restrictions $A(0) = A'(0) = 0$ have now been placed upon $A(\xi)$.

The evaluation of the inverse transform of $\bar{\Omega}(s)$ is given in the appendix. It is shown that over the range of $0 \leq \xi \leq 2$ the relation

$$\Omega(\xi) = -\frac{1}{\pi}\ln\frac{2-\xi}{2} + T(\xi) \tag{35}$$

holds, where $T(\xi)$ is a power series that is convergent over the range. Sketch (i) shows $\Omega(\xi)$, and the accompanying table lists values of the function and its component parts for $0 \leq \xi \leq 2$.



Sketch (i)

ξ	$-\frac{1}{\pi}\ln\frac{2-\xi}{2}$	$T(\xi)$	$\Omega(\xi)$
0.20	0.033538	0.122638	0.156176
.40	.071029	.246073	.317102
.60	.113534	.403432	.516966
.80	.162601	.568452	.731053
1.00	.220636	.755683	.976319
1.20	.291665	.970676	1.262341
1.40	.383237	1.220196	1.603433
1.60	.512300	1.512421	2.024721
1.80	.732936	1.857148	2.590084
1.95	1.174207	2.157066	3.331273
2.00	∞	2.266000	∞

Equations (33) now must be substituted into equation (31). The resultant expression is the drag of the configuration. The linearity of the term $\partial\varphi/\partial\xi$ in the integrand, moreover, permits one to write drag in the form $D = D_{bb} + D_{bi}$ where D_{bb} is the drag of the body as it exists alone in its own induced flow field and D_{bi} is the drag of the body attributable to the induced effects of the shrouding tube.

Up to this point in the analysis no explicit use has been made of the assumption that the body closes and, in fact, straightforward evaluation of D_{bb} leads to the expression given by Ward (ref. 4) and Frankl and Karpovich (ref. 7) for open-ended bodies with finite slopes at the base. If, however, the body closes or has zero slope at the end, there results the simpler drag formula of von Kármán (ref. 8).

$$\begin{aligned}\frac{D_{bb}}{q_o} &= -\frac{h^2}{2\pi}\int_0^{2/m} A''(\xi)d\xi\int_0^{2/m} A''(\xi_1)\ln|\xi - \xi_1|d\xi_1 \\ &= \frac{h^2}{2\pi}\int_0^{2/m}\int_0^{2/m} A'(\xi)A''(\xi_1)\frac{d\xi\, d\xi_1}{\xi - \xi_1}\end{aligned} \tag{36}$$

The interference drag is then

$$\begin{aligned}\frac{D_{bi}}{q_o} &= \beta_o h^2\int_2^{2/m}\left[-\frac{2}{\beta_o}\frac{\partial\varphi_2}{\partial\xi} - \left(\frac{\partial\varphi_2}{\partial\eta}\right)^2\right]_b\frac{dA}{d\xi}d\xi \\ &= -h^2\int_2^{2/m} 2\frac{\partial\varphi_2}{\partial\xi}\frac{dA}{d\xi}d\xi\end{aligned}$$

where $\partial\varphi_2/\partial\xi$ is given by the last two terms in the right member of equation (33b). Substitution into the equation for the interference drag yields

$$\frac{D_{bi}}{q_o} = h^2\int_2^{2/m} A'(\xi)\left[A'''(\xi - 2) + A''(0)\Omega(\xi - 2) + \int_2^{\xi} A'''(\xi_1 - 2)\Omega(\xi - \xi_1)d\xi_1\right]d\xi \tag{37}$$

so that the formula for the total wave drag for a closed body of arbitrary shape becomes

$$\left.\begin{aligned}\frac{D}{q_o} &= h^2\int_2^{2/m} A'(\xi)\left[A''(\xi - 2) + A''(0)\Omega(\xi - 2) + \int_2^{\xi} A'''(\xi_1 - 2)\Omega(\xi - \xi_1)d\xi_1\right]d\xi\ + \\ &\quad \frac{h^2}{2\pi}\int_0^{2/m} A'(\xi)d\xi\int_0^{2/m}\frac{A''(\xi_1)}{\xi - \xi_1}d\xi_1, \qquad 1/2 \le m \le 1 \\ \frac{D}{q_o} &= \frac{h^2}{2\pi}\int_0^{2/m} A'(\xi)d\xi\int_0^{2/m}\frac{A''(\xi_1)}{\xi - \xi_1}d\xi_1, \qquad m > 1\end{aligned}\right\} \tag{38a}$$

In terms of the physical variables, using equation (25), one may write the drag in the final form

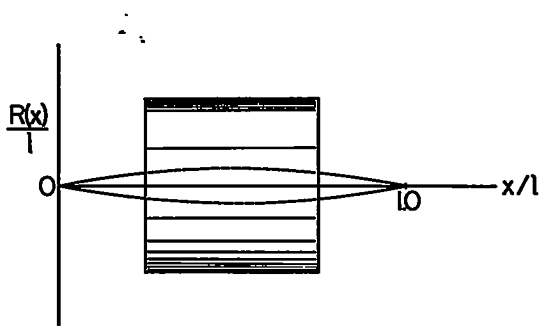
$$
\left.
\begin{aligned}
\frac{D}{q_o} &= \int_{ml}^{l} S'(x)\left[S''(x-ml)+S'''(0)\Omega\left(\frac{2x}{ml}-2\right)+\int_0^{x-ml} S'''(x-ml-x_1)\Omega\left(\frac{2x_1}{ml}\right)dx_1\right]dx + \\
&\quad \frac{1}{2\pi}\int_0^{l} S'(x)dx\int_0^{l}\frac{S''(x_1)}{x-x_1}\,dx_1\,, \qquad 1/2 \le m \le 1 \\
\frac{D}{q_o} &= \frac{1}{2\pi}\int_0^{l} S'(x)dx\int_0^{l}\frac{S''(x_1)}{x-x_1}\,dx_1\,, \qquad m > 1
\end{aligned}
\right\} \tag{38b}
$$

where $m = 2\beta_o h/l$. If the shrouding tube of the configuration is designed for a fixed radius h, formulas (38) give the drag for a shrouded body of revolution of arbitrary shape for Mach numbers $M_o \ge \sqrt{1+l^2/16h^2}$. When $m = 1/2$ the waves from the forward portion of the body are reflected onto the rearward as in sketch (a). For $m \ge 1$ there is no effect of the tube and the formula for the drag is the same as that of the body without shroud.

As an example, consider the body of revolution whose generating curve is defined by

$$R(x) = \frac{2t_o}{l^2}\,x(l-x)\,, \qquad 0 \le x \le l \tag{39a}$$

where t_o is the maximum thickness (see sketch (j)). Here



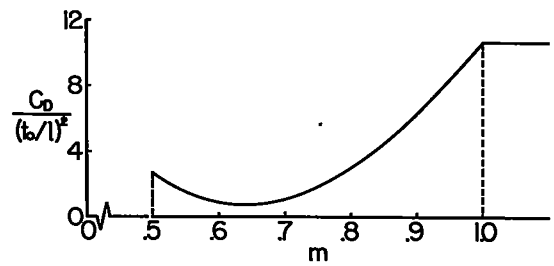
Sketch (j)

$$
\left.
\begin{aligned}
S(x) &= \frac{4\pi t_o^2}{l^4}\,x^2(l-x)^2 \\
S'(x) &= \frac{8\pi t_o^2}{l^4}\,x(l-x)(l-2x) \\
S''(x) &= \frac{8\pi t_o^2}{l^4}\,(l^2-6xl+6x^2) \\
S'''(x) &= \frac{48\pi t_o^2}{l^4}\,(2x-l)
\end{aligned}
\right\} \tag{39b}
$$

with frontal area $S_F = \pi t_o^2/4$. For $m > 1$ the second formula in equations (38b) yields

$$C_D = \frac{D}{q_o S_f} = \frac{32}{3}\left(\frac{t_o}{l}\right)^2 \qquad (40)$$

Integration of the first relation in formulas (38b) for $1/2 \leq m \leq 1$ is carried out numerically and the results are shown in sketch (k). For a fixed value of the ratio h/l, this sketch gives the drag of the configuration as a function of β_o for $\beta_o \geq l/4h$.



Sketch (k)

Shrouded Bodies Having Zero Wave Drag

In general, the value found from the drag formula (38) for a shrouded body will be greater than zero. It is possible to obtain from this formula, however, an integral equation whose solution will yield a class of body shapes for which the wave drag is zero.[1]

Derivation of integral equation.- In the particular application to be considered here, we take $\beta_o h = l/4$ $(m = 1/2)$ and assume that $A''(0)=0$. A further assumption to be made is that the body has fore-and-aft symmetry. This implies that $A(\xi) = A(4-\xi)$ and, since no discontinuities are allowed in the slope of the meridian section, $A'(2) = 0$. Under these conditions the first relation in equation (38a) becomes

$$\frac{D}{q_o} = h^2\int_2^4 A'(\xi)\left[A''(\xi-2)+\int_2^{\xi} A'''(\xi_1-2)\Omega(\xi-\xi_1)d\xi_1+\frac{1}{\pi}\int_0^4 \frac{A''(\xi_1)}{\xi-\xi_1}\,d\xi_1\right]d\xi \tag{41}$$

A sufficient condition for the drag to vanish is that the expression within the brackets be zero, resulting in the relation

$$A''(\xi-2)+\int_2^{\xi} A'''(\xi_1-2)\Omega(\xi-\xi_1)d\xi_1+\frac{1}{\pi}\int_0^4 \frac{A''(\xi_1)}{\xi-\xi_1}\,d\xi_1 = 0\ , \qquad 2 \leq \xi \leq 4 \tag{42a}$$

The solution of this integral equation for the function $A(\xi)$ is required in order to find the area distribution $S(x) = \beta_o h^2 A(\xi)$ of the body whose drag is canceled by the shrouding tube.

[1]While this part of the analysis was being carried out, Graham, Beane, and Licher (ref. 9) published a paper treating essentially the same problem from a different point of view.

Solution of integral equation.- Equation (42a) can also be written

$$E(\xi) = A''(\xi) + \int_0^\xi A''(\xi_1)\psi(\xi - \xi_1)d\xi_1 , \qquad 0 \leq \xi \leq 2 \tag{42b}$$

where

$$\left.\begin{aligned} E(\xi) &= -\frac{1}{\pi}\int_0^2\left(\frac{1}{\xi - 2 - \xi_1} + \frac{1}{\xi - 2 + \xi_1}\right)A''(\xi_1)d\xi_1 \\ \psi(\xi) &= \frac{d}{d\xi}\,\Omega(\xi) \end{aligned}\right\} \tag{43}$$

In dealing with this integral equation it is advantageous to apply the calculus of Laplace transform because the equation contains the convolution of A'' and ψ.

Taking the Laplace transform of both sides of relation (42b), one obtains

$$\bar{E}(s) = s^2\bar{A}(s)[1 + s\bar{\Omega}(s)] \tag{44a}$$

or

$$\bar{E}(s) = s^2\bar{A}(s)\,\frac{e^s K_1(s)}{\pi e^{-s} I_1(s)} \tag{44b}$$

For a homogeneous equation of the type (42b) it is usually convenient to assume that the unknown can be expressed in the form

$$A''(\xi) = \int_0^\xi \gamma(\xi_2)g(\xi - \xi_2)d\xi_2 \tag{45}$$

where the functions g and γ are to be determined. Equation (44b) then yields

$$\bar{E}(s) = \bar{\gamma}(s)\bar{g}(s)\,\frac{e^s K_1(s)}{\pi e^{-s} I_1(s)} \tag{46}$$

Now the quantity in the denominator of this relation can be canceled by simply assuming

$$\bar{\gamma}(s) = \pi e^{-s} I_1(s) \quad . \tag{47a}$$

or

$$\left.\begin{aligned} \gamma(\xi_2) &= \frac{1 - \xi_2}{\sqrt{\xi_2(2 - \xi_2)}} , \qquad 0 \le \xi_2 \le 2 \\ &= 0 , \qquad\qquad\qquad \xi_2 > 2 \end{aligned}\right\} \tag{47b}$$

so that equation (46) reduces to

$$\bar{E}(s) = \bar{g}(s)e^{s}K_1(s) \tag{48a}$$

or

$$E(\xi) = \int_0^{\xi} \frac{1 + \xi_1}{\sqrt{\xi_1(2 + \xi_1)}} g(\xi - \xi_1)d\xi_1 \tag{48b}$$

From equations (45), (47b), (48b), and the first relation in equations (43), there results the following integral equation for determining the function g:

$$\int_0^{\xi} \frac{1 + \xi_1}{\sqrt{\xi_1(2 + \xi_1)}} g(\xi - \xi_1)d\xi_1$$

$$= - \frac{1}{\pi}\int_0^{2}\left[\frac{1}{\xi - 2 - \xi_1} + \frac{1}{\xi - 2 + \xi_1}\right]d\xi_1\int_0^{\xi_1} \frac{1 - \xi_2}{\sqrt{\xi_2(2 - \xi_2)}} g(\xi_1 - \xi_2)d\xi_2 \tag{49}$$

After integration and some manipulation, relation (49) reduces to the homogeneous equation of the first kind

$$\int_0^{2} G_1(\xi_1)k(\xi,\xi_1)d\xi_1 = 0 , \qquad 0 \le \xi \le 2 \tag{50}$$

with unknown

$$G_1(\xi_1) = g(\xi_1) + g(2 - \xi_1) \tag{51}$$

and kernel

$$k(\xi,\xi_1) = \frac{1 + \xi - \xi_1}{\sqrt{(\xi - \xi_1)(2 + \xi - \xi_1)}} \tan^{-1}\sqrt{\frac{(2 - \xi_1)(\xi - \xi_1)}{(2 + \xi - \xi_1)\xi_1}} \tag{52}$$

This expression obviously has at least the trivial solution $G_1(\xi_1) \equiv 0$, or

$$g(\xi_1) = -\,g(2-\xi_1)\ , \qquad 0 \le \xi_1 \le 2 \tag{53}$$

requiring simply that the function g in equation (45) be odd about $\xi_1 = 1$. For a function that satisfies this condition, it follows that a formal solution of the integral equation (42) can be written as

$$A''(\xi) = \int_0^{\xi} \frac{1-\xi_1}{\sqrt{\xi_1(2-\xi_1)}}\, g(\xi-\xi_1)d\xi_1\ , \qquad 0 \le \xi \le 2 \tag{54a}$$

or

$$A'(\xi) = \int_0^{\xi} \sqrt{\xi_1(2-\xi_1)}\, g(\xi-\xi_1)d\xi_1 \tag{54b}$$

In order to avoid obtaining bodies with negative areas, however, the function g satisfying relation (53) must be chosen so that

$$A(\xi) = \frac{1}{2}\int_0^{\xi}\left[(\xi_1-1)\sqrt{\xi_1(2-\xi_1)} + \cos^{-1}(1-\xi_1)\right] g(\xi-\xi_1)d\xi_1 \tag{54c}$$

is greater than or equal to zero. These formulas determine the shape of the shrouded body for the range $0 \le \xi \le 2$, while the rear half of the body is obtained by reflection about $\xi = 2$. There is of course an infinitude of functions for which the relation (53) holds, so that equations (54) actually furnish an infinite class of body shapes whose drag is canceled by a shrouding cylindrical shell. It will be shown later that this function g can be related to the pressure coefficient on the shroud.

The quantities defined by formulas (54) are continuous if g is piecewise continuous. It is apparent from these equations that the restriction $A''(0) = A'(0) = A(0) = 0$ is met, and it can be seen by use of equation (53) that the requirement $A'(2) = 0$ is also satisfied. Moreover, it can be shown that if A' is required to vanish at $\xi = 2$, and if g is restricted to be piecewise continuous, the solution $G_1(\xi_1) \equiv 0$ used for equation (50) is the only one admissible here.

A simple function that satisfies the functional relation (53) is

$$\left.\begin{aligned} g(\xi_1) &= \lambda_2\xi_1\ , && 0 \le \xi_1 \le 1 \\ &= \lambda_2(\xi_1-2)\ , && 1 \le \xi_1 \le 2 \end{aligned}\right\} \tag{55}$$

Inserting this expression into the formula (54c) yields, for $1 \le \xi \le 2$,

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THEORETICAL WAVE DRAG OF SHROUDED AIRFOILS AND BODIES

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SUMMARY

Formulas for the wave drag of shrouded symmetrical airfoils and shrouded bodies of revolution of arbitrary shape are derived by means of linearized theory. In the case of the airfoils the shroud consists of flat plates, and for the bodies of revolution the shroud is a cylindrical shell. The results obtained hold for a Mach number range dependent on the geometry of the configuration. Expressions are also given for determining a class of body shapes for which the wave drag is theoretically zero.

INTRODUCTION

A body moving at supersonic speeds has a wave drag which can be calculated either from integrations based upon the pressure at the surface of the body or by means of a momentum balance over a control surface surrounding the body. The control-surface approach shows more clearly that the wave drag is related to the transport of momentum in the Mach waves created by the body. This approach also suggests the scheme of reducing or destroying the wave drag through the use of a shroud as first shown by Ferrari (ref. 1). With such a shroud the waves are caught and reflected to the body surfaces where they may be absorbed without further reflection. From the standpoint of the pressure exerted on the body itself, it follows that the reflected waves may strike the rear portion of the body in such a way as to provide a buoyancy to overcome the resistance of the body alone. The detrimental effect of the additional friction drag due to a shroud is not included in the present study.

The principal object of the present investigation is to derive formulas for the wave drag of shrouded symmetrical airfoils and shrouded bodies of revolution of arbitrary shape. The airfoil is shrouded by flat plates and the body of revolution is shrouded by a cylindrical shell. Although many configurations are possible, the analysis here considers the particular arrangement where the shroud extends at least far enough forward to catch the Mach wave emanating from the body nose, and far enough rearward to cast Mach waves on the base of the body. As a special application of the results obtained, a class of body shapes, similar to those given by Busemann (ref. 2) and Ferri (ref. 3), are found for which the wave drag is theoretically zero.

For either the case of the airfoil or the body of revolution, the analysis is based on the assumption of linearized theory and on operational methods using Laplace transforms. Ward (refs. 4 and 5) has shown in considerable detail how operational calculus can be employed to treat bodies of revolution and quasi-cylindrical tubes. A similar approach will be followed here, except that Heaviside notation is not used.

To the order of the analysis employed, discontinuities in the slope of the airfoil are admitted, while for the body of revolution the gradient of cross-sectional area and its derivative are assumed to undergo no abrupt changes. In actual practice, discontinuities producing fixed compression waves would certainly upset the accuracy of the results even more than for the airfoil or the body alone since the opposing surfaces offer the possibility of shock-wave and boundary-layer interaction.

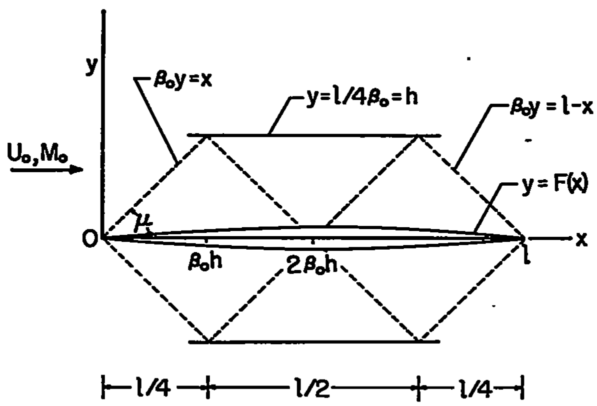
SYMBOLS

a_o	speed of sound in free stream
A,B	cross-sectional area of body in dimensionless terms, $\frac{S}{\beta_o h^2}$
c_d	section drag coefficient, $\frac{D}{q_o l}$
C_D	body drag coefficient, $\frac{D}{q_o S_f}$
C_p	pressure coefficient, $\frac{p - p_o}{q_o}$
D	wave drag
f	function defined in equation (5), $\frac{F}{\beta_o h}$
F	function defining upper surface of airfoil or generating curve of the body of revolution
g	function satisfying relation (53)
h	distance of shroud from x axis

$I_0, I_1, J_0, J_1, K_0, K_1$	Bessel functions (ref. 6)
l	length of body, or chord length of airfoil
m	$\frac{2\beta_0 h}{l}$
M_0	Mach number in the free stream
p	local pressure
p_0	pressure in free stream
q_0	free-stream dynamic pressure, $\frac{\rho_0 U_0^2}{2}$
R	radius of body
$S(x)$	cross-sectional area of body
S_f	frontal area of body
t_0	maximum thickness of body
U_0	free-stream velocity
x	Cartesian coordinate in free-stream direction
y	Cartesian coordinate, measuring vertical distance for airfoil and radial distance for body
Y_1	Bessel function (ref. 6)
β_0	$\sqrt{M_0^2 - 1}$
η	dimensionless variable introduced in equation (4), $\frac{y}{h}$
μ	Mach angle, arc sin $\frac{1}{M_0}$
ξ	dimensionless variable introduced in equation (4), $\frac{x}{\beta_0 h}$
ρ_0	free-stream density
$\varphi(\xi,\eta)$	perturbation velocity potential in dimensionless terms, $\frac{\Phi(x,y)}{U_0 h}$

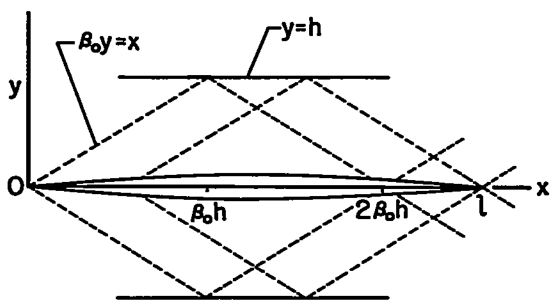
$\phi(x,y)$	perturbation velocity potential
Ω	influence function defined in equation (35)
$(\bar{\ })$	Laplace transform of function

MATHEMATICAL STATEMENT OF PROBLEM



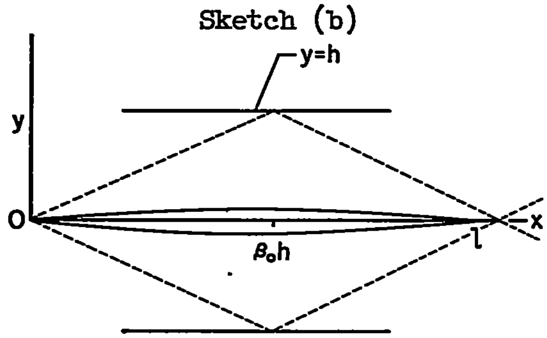
($\beta_0 h = l/4$)

Sketch (a)



($l/4 < \beta_0 h < l/2$)

Sketch (b)



($\beta_0 h > l/2$)

Sketch (c)

Consider a symmetrical airfoil or slender body of revolution placed at zero angle of attack at Mach number $M_o = U_o/a_o > 1$, where a_o is the speed of sound in the free stream and where U_o, the free-stream velocity, is alined with the x axis. The y axis measures vertical distance in the case of the airfoil and radial distance in the case of the body. The nose of the body is at the origin of the coordinate system.

In the configurations to be considered the airfoil is shrouded by two flat plates and the body of revolution is shrouded by a cylindrical tube. The shrouding plates (or tubes) are placed so that the distance h by which the shroud is removed from the x axis is such that $l/4 \leq \beta_o h \leq l/2$, where $\beta_o = \cot \mu = \sqrt{M_o^2 - 1}$ and l is the body length. The shroud is required to extend at least from $x = \beta_o h$ to $x = l - \beta_o h$, but it would produce no additional effects on the body if it were longer. Sketches (a), (b), and (c) show the geometry of the configuration with either the airfoil or body of revolution in three typical arrangements. If $\beta_o h = l/4$, all the Mach waves from the forward portion of the body are reflected onto the rearward, and when $\beta_o h \geq l/2$ there is no effect of the shroud since the waves are reflected behind the body.

The upper surface (or generating curve) of the airfoil or body is assumed given by the function

$$y = F(x) \tag{1}$$

It will also be assumed that the body closes at both ends (i.e., that $F(0) = F(l) = 0$) and that the thickness-length ratio of the body is sufficiently small relative to the Mach angle μ that linearized theory applies. As a result, the perturbation velocity potential $\Phi(x,y)$ satisfies the partial differential equation

$$\beta_o{}^2 \frac{\partial^2\Phi}{\partial x^2} - \frac{\partial^2\Phi}{\partial y^2} - \frac{\nu}{y}\frac{\partial\Phi}{\partial y} = 0 \tag{2}$$

together with the boundary conditions

$$\left.\begin{aligned} \left(\frac{y^\nu}{U_o}\frac{\partial\Phi}{\partial y}\right)_{y=\nu F} &= F^\nu \frac{dF}{dx}\,, \qquad 0 \le x \le l \\ \left(\frac{1}{U_o}\frac{\partial\Phi}{\partial y}\right)_{y=h} &= 0\,, \qquad \beta_o h \le x \le l - \beta_o h \end{aligned}\right\} \tag{3}$$

where the parameter ν equals 0 and 1, respectively, for the airfoil and body of revolution. One then has the problem of finding a solution of Φ, and from this to determine the drag of the configuration.

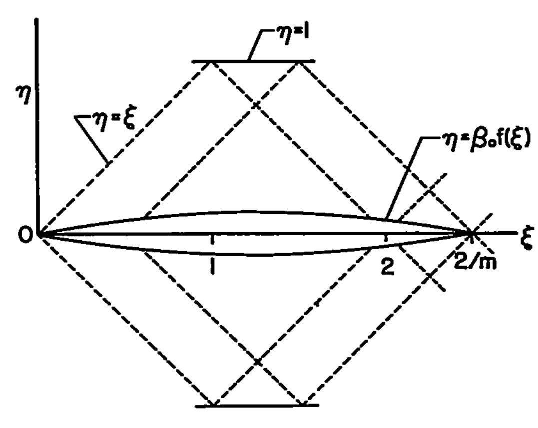
If dimensionless variables ξ, η, φ defined by the relations

$$\xi = x/\beta_o h\,, \qquad \eta = y/h\,, \qquad \varphi(\xi,\eta) = \Phi(x,y)/U_o h \tag{4}$$

are now introduced, the expression for the body surface, as given by equation (1), becomes

$$\eta = \frac{2\beta_o}{ml} F\left(\frac{ml}{2}\,\xi\right) = \beta_o f(\xi) \tag{5}$$

with $m = 2\beta_o h/l$, and sketch (b) becomes sketch (d). The differential equation is then



(1/2 < m < 1)

Sketch (d)

$$\frac{\partial^2\varphi}{\partial\xi^2} - \frac{\partial^2\varphi}{\partial\eta^2} - \frac{\nu}{\eta}\frac{\partial\varphi}{\partial\eta} = 0 \tag{6}$$

with boundary conditions

$$\left.\begin{aligned} \left(\eta^\nu \frac{\partial\varphi}{\partial\eta}\right)_{\eta=\nu\beta_o f} &= (\beta_o f)^\nu \frac{df}{d\xi}, & 0 \leq \xi \leq 2/m \\ \left(\frac{\partial\varphi}{\partial\eta}\right)_{\eta=1} &= 0, & 1 \leq \xi \leq 2/m - 1 \end{aligned}\right\} \tag{7}$$

Operational methods based upon the Laplace transform are suited for treating the basic differential equation (6) for either the airfoil or the body of revolution. Denoting the Laplace transform of a function $g(\xi,\eta)$ by $\bar{g}(s;\eta)$ where

$$\bar{g}(s;\eta) = \int_0^\infty e^{-s\xi} g(\xi,\eta)\,d\xi \tag{8}$$

and employing this transformation for relations (6) and (7), one obtains the differential equation

$$s^2\bar{\varphi} - \frac{\partial^2\bar{\varphi}}{\partial\eta^2} - \frac{\nu}{\eta}\frac{\partial\bar{\varphi}}{\partial\eta} = 0 \tag{9}$$

together with the boundary conditions

$$\left(\eta^\nu \frac{\partial\bar{\varphi}}{\partial\eta}\right)_{\eta=\nu\beta_o f} = (\beta_o)^\nu\, \overline{f^\nu \frac{df}{d\xi}} \tag{10a}$$

and

$$\left(\frac{\partial\bar{\varphi}}{\partial\eta}\right)_{\eta=1} = 0 \tag{10b}$$

Once $\bar{\varphi}$ has been determined from equation (9), the drag of the shrouded figures can be calculated. In order to carry out the calculations, it is convenient to treat the airfoil and body-of-revolution problems separately. The analysis for the airfoil offers little in the way of novelty but will be given first since it illustrates the framework of the methods employed.

AIRFOIL WITH SHROUDING PLATES

Evaluation of Wave Drag

The solution of equation (9) for the case of the airfoil (i.e., when $\nu = 0$) is

$$\bar{\varphi}(s;\eta) = a(s)e^{-s\eta} + b(s)e^{s\eta} \tag{11}$$

Since vertical symmetry exists in the flow field, attention can be limited to the upper half of the figure. Over the forward part of the airfoil the downstream inclination of the outgoing waves indicates, furthermore, that the second term in the answer (11) can be deleted. Imposing the boundary condition (10a), one then has

$$-sa(s)e^{-s\eta}\Big]_{\eta=0} = s\bar{f} \ , \qquad a(s) = -\bar{f}$$

and it follows that the perturbation velocity potential satisfies the relations

$$\left.\begin{aligned} \bar{\varphi}(s;\eta) &= -\bar{f}e^{-s\eta} \\ \varphi(\xi,\eta) &= -f(\xi - \eta) \end{aligned}\right\} \tag{12a}$$

or, in terms of the physical variables,

$$\Phi(x,y) = -\frac{U_0}{\beta_0}F(x - \beta_0 y) \ , \qquad 0 \leq x - \beta_0 y \leq ml - 2\beta_0 y \tag{12b}$$

where $m = 2\beta_0 h/l$, $1/2 \leq m \leq 1$.

The flow around the forward portion of the airfoil is given by equations (12); in order to predict the flow around the rearward portion, however, it is necessary to determine the nature of the incoming waves from the plate. The velocity potential of these incoming waves can be obtained from the second term in the right member of equation (11). Boundary condition (10b) requires that vertical velocity be zero at $\eta = 1$. From equations (12) and (11), therefore, one has

$$sb(s)e^{s\eta}\Big]_{\eta=1} = -s\bar{f}e^{-s\eta}\Big]_{\eta=1} ; \qquad b(s) = -\bar{f}e^{-2s}$$

Hence the potential φ_2 of the incoming waves is determined by

$$\left.\begin{aligned}\bar{\varphi}_2(s;\eta) &= -\bar{f}e^{-s(2-\eta)}\\ \varphi_2(\xi,\eta) &= -f(\xi+\eta-2)\end{aligned}\right\} \tag{13}$$

and the potential in the region of the plate is

$$\varphi = -f(\xi-\eta) - f(\xi+\eta-2)$$

If the potential φ over the rear of the airfoil is written in the form

$$\varphi(\xi,\eta) = \varphi_2(\xi,\eta) + \varphi_3(\xi,\eta)$$

and $\bar{\varphi}_3$ is assumed expressible by terms of the form $a_3(s)e^{-s\eta}$, the boundary condition (10a) yields

$$-s\bar{f}e^{-s(2-\eta)}\Big]_{\eta=0} - sa_3(s)e^{-s\eta}\Big]_{\eta=0} = s\bar{f}$$

Thus $a_3(s) = -\bar{f}e^{-2s} - \bar{f}$, so that

$$\varphi_3(\xi,\eta) = -f(\xi-\eta-2) - f(\xi-\eta)$$

The potential over the rear of the airfoil is then given as follows:

$$\varphi(\xi,\eta) = -f(\xi-\eta-2) - f(\xi+\eta-2) - f(\xi-\eta) \tag{14a}$$

or finally

$$\Phi(x,y) = -\frac{U_o}{\beta_o}\left[F(x-\beta_o y-ml) + F(x+\beta_o y-ml) + F(x-\beta_o y)\right], \quad ml \leq x-\beta_o y \leq l \tag{14b}$$

It remains now to find the expression for wave drag on the surface. In the physical variables, the total wave drag is

$$\frac{D}{q_o} = 2\int_0^l \left(\frac{p-p_o}{q_o}\right)_{y=0} \frac{dF}{dx}\,dx = 2\int_0^l C_p(x,o)\,\frac{dF}{dx}\,dx \tag{15}$$

where $q_o = \rho_o U_o{}^2/2$ is the dynamic pressure in the free stream of density ρ_o. With the substitution, from thin-airfoil theory, $C_p = -(2/U_o)(\partial\Phi/\partial x)$, the pressure coefficient on the airfoil surface is

$$\left.\begin{aligned} C_p &= \frac{2}{\beta_o} F'(x) \ , & 0 \leq x \leq ml \\ C_p &= \frac{2}{\beta_o} [F'(x) + 2F'(x - ml)] \ , & ml \leq x \leq l \\ & & (1/2 \leq m \leq 1) \end{aligned}\right\}$$

and the drag, expressed in coefficient form, can be written as

$$\left.\begin{aligned} c_d &= \frac{4}{\beta_o l}\int_{ml}^{l} [F'(x) + F'(x - ml)]^2 dx + \frac{4}{\beta_o l}\int_{(1-m)l}^{l} [F'(x)]^2 dx \ , & 1/2 \leq m \leq 1 \\ c_d &= \frac{4}{\beta_o l}\int_0^{l} [F'(x)]^2 dx \ , & m > 1 \end{aligned}\right\} \tag{16}$$

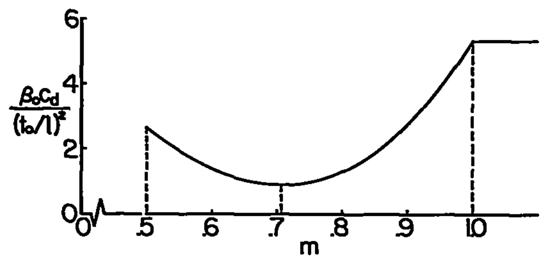
If the reflecting plates of the configuration are designed for a fixed height h, formulas (16) give the drag coefficient of a shrouded airfoil of arbitrary shape for Mach numbers $M_o \geq \sqrt{1 + l^2/16h^2}$. When $\beta_o h = l/4$ $(m = 1/2)$, the waves are reflected from the forward portion of the airfoil onto the rearward as in sketch (a), and for $m > 1$ the reflected waves do not intersect the airfoil. Formulas (16) thus yield results for an airfoil with or without shroud.

Consider, as a simple example, the shrouded biconvex airfoil section whose upper surface is defined by

$$F(x) = \frac{2t_o}{l^2} x(l - x) \ , \qquad 0 \leq x \leq l \tag{17}$$

where t_o is the maximum thickness of the airfoil. The drag coefficient, given by formulas (16), is then

$$\left.\begin{aligned} c_d &= \frac{16}{3\beta_o}\left(\frac{t_o}{l}\right)^2(4m^3 - 6m + 3)\ , \qquad 1/2 \leq m \leq 1 \\ c_d &= \frac{16}{3\beta_o}\left(\frac{t_o}{l}\right)^2\ , \qquad m > 1 \end{aligned}\right\} \tag{18}$$



Sketch (e)

with the parameter $m = 2\beta_o h/l$. It is seen in sketch (e) that for this special case, the value of $\beta_o c_d$ for $1/2 \leq m \leq 1$ is always less than its value for $m \geq 1$, and that a relative minimum occurs at $m = 1/\sqrt{2}$. If the ratio h/l is fixed, formulas (18) give the drag of the shrouded airfoil as a function of β_o for $\beta_o \geq l/4h$.

Shrouded Airfoils Having Zero Wave Drag

Determination of shape.- Although formula (16) for the drag coefficient will in general be greater than zero, there exist classes of airfoil shapes for which the drag is theoretically zero. Since the two integrals in the first expression in equations (16) can never be negative, the necessary and sufficient condition that the drag vanish is that

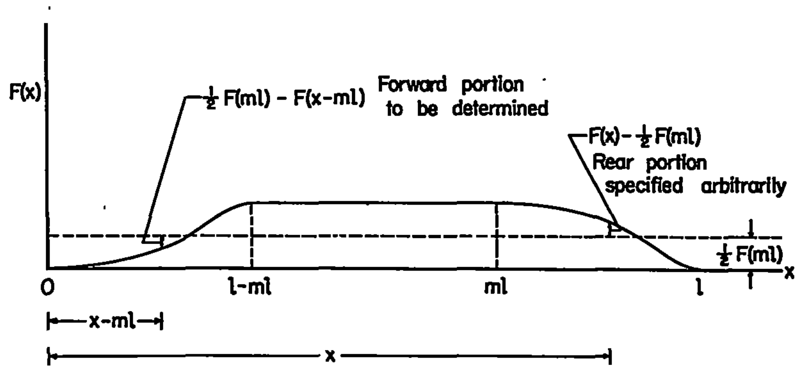
$$\left.\begin{aligned} F'(x) &= -F'(x - ml)\ , \qquad ml \leq x \leq l \\ F'(x) &= 0\ , \qquad (1 - m)l \leq x \leq ml \end{aligned}\right\} \tag{19a}$$

After integration, these relations become

$$\left.\begin{aligned} \frac{1}{2}F(ml) - F(x - ml) &= F(x) - \frac{1}{2}F(ml)\ , \qquad ml \leq x \leq l \\ F(x) &= F(ml)\ , \qquad (1 - m)l \leq x \leq ml \end{aligned}\right\} \tag{19b}$$

Thus the airfoil can be drawn in an arbitrary manner from $x = ml$ to $x = l$ and the forward portion of the airfoil shape in the interval $0 \leq x \leq (1 - m)l$ is determined; the portion in the interval

$(1 - m)l \leq x \leq ml$ is flat and equal to $F(ml)$. Sketch (f) shows the geometrical construction of the profile. The upper half of such an airfoil is equivalent to the lower wing of the linearized version of a Busemann biplane arrangement, while the lower half is the upper wing.



Sketch (f)

For the special case when $m = 1/2$, as in sketch (a), equations (19b) can be written in the form

$$F(x) - \frac{1}{2}F\left(\frac{l}{2}\right) = \frac{1}{2}F\left(\frac{l}{2}\right) - F\left(x - \frac{l}{2}\right), \qquad l/2 \leq x \leq l \tag{19c}$$

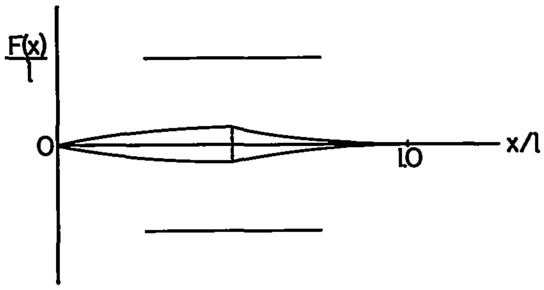
If, moreover, the airfoil is assumed to be symmetrical fore and aft, $F(x) = F(l - x)$ and equation (19c) becomes

$$F(x) - \frac{1}{2}F\left(\frac{l}{2}\right) = \frac{1}{2}F\left(\frac{l}{2}\right) - F\left(\frac{l}{2} - x\right), \qquad 0 \leq x \leq l/2 \tag{19d}$$

In this event the forward half of the airfoil has odd symmetry about the ordinate of the quarter-chord position and, similarly, the rear half has odd symmetry about the three-quarter chord position. It is also found that the pressure distribution on the airfoil has fore-and-aft symmetry.

Drag for off-design condition.- A shrouded airfoil whose geometry satisfies relation (19a), however, will only have zero drag for some particular value of the parameter m, say m_o. If such an airfoil is moving so that the parameter m is different from m_o, formulas (16) for the drag coefficient become

$$\left.\begin{aligned}
c_d &= \frac{4}{\beta_o l}\int_0^l [F'(x)]^2 dx - \frac{8}{\beta_o l}\int_{m_o l}^{l} F'(x - m_o l)F'(x - ml)dx, && 1/2 \leq m \leq m_o \leq 1 \\
c_d &= \frac{4}{\beta_o l}\int_0^l [F'(x)]^2 dx - \frac{8}{\beta_o l}\int_{ml}^{l} F'(x - m_o l)F'(x - ml)dx, && 1/2 \leq m_o \leq m \leq 1 \\
c_d &= \frac{4}{\beta_o l}\int_0^l [F'(x)]^2 dx, && m > 1
\end{aligned}\right\} \tag{20}$$



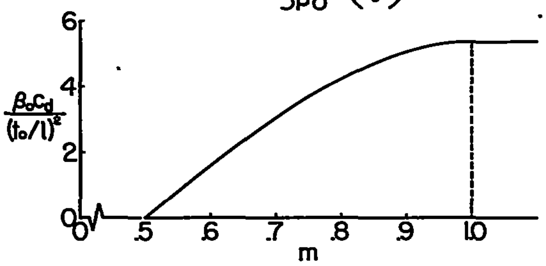
Sketch (g)

As an example, let the upper surface of the airfoil (shown in sketch (g)) be defined by

$$\left.\begin{aligned} F(x) &= \frac{2t_o}{l^2}\,x(l - x)\ , & 0 \le x \le l/2 \\ &= \frac{2t_o}{l^2}\,(l - x)^2 , & l/2 \le x \le l \end{aligned}\right\} \tag{21}$$

If the design of the configuration is such that $2\beta_o h = l/2$ (i.e., $m_o = 1/2$), relation (19a) is satisfied so that there is no drag. When $l/2 \le 2\beta_o h \le l$, however, the drag coefficient for the airfoil is given, from equation (20), by

$$\left.\begin{aligned} c_d &= \frac{16}{3\beta_o}\left(\frac{t_o}{l}\right)^2 (-4m^3 + 6m^2 - 1)\ , & 1/2 \le m \le 1 \\ c_d &= \frac{16}{3\beta_o}\left(\frac{t_o}{l}\right)^2 , & m > 1 \end{aligned}\right\} \tag{22}$$



Sketch (h)

A plot of $\beta_o c_d/(t_o/l)^2$ against m is shown in sketch (h) for values of m greater than 1/2. Below $m = 1/2$ the calculations become more involved but the same general method is applicable.

BODY OF REVOLUTION WITH SHROUDING TUBE

Evaluation of Wave Drag

The solution of equation (9) for the case of the body of revolution (i.e., when $\nu = 1$) is

$$\bar{\varphi}(s;\eta) = a(s)K_o(s\eta) + b(s)I_o(s\eta) \tag{23}$$

where Watson's notation (ref. 6) for the Bessel functions K_o and I_o is used. Over the forward portion of the body the wave system is outgoing and the solution can be formulated from the first term in the right member of equation (23). In order to impose the boundary condition (10a), one employs the relation

$$\frac{\partial\bar{\varphi}}{\partial\eta} = -\, sa(s)K_1(s\eta) \simeq -\frac{a(s)}{\eta}$$

the last expression holding by virtue of the fact that η is small on the body surface. The boundary condition then yields, for $f(0) = 0$,

$$a(s) = -\frac{s\bar{A}(s)}{2\pi} \tag{24}$$

where the quantity A is

$$A(\xi) = \beta_o\pi f^2(\xi) = \frac{\pi F^2(x)}{\beta_o h^2} = \frac{\pi R^2(x)}{\beta_o h^2} = \frac{S(x)}{\beta_o h^2} \tag{25}$$

$S(x)$ being the cross-sectional area of the body. The solution over the forward portion of the body is thus

$$\bar{\varphi}(s;\eta) = -\frac{1}{2\pi}\, s\bar{A}(s)K_o(s\eta) \tag{26}$$

From equation (26) and the boundary condition (10b), the potential φ_2 of the incoming waves from the shrouding tube can be calculated. If the Laplace transform of φ_2 is assumed expressible in the form

$$\bar{\varphi}_2(s;\eta) = b(s)I_o(s\eta)\ , \qquad \frac{\partial\bar{\varphi}_2}{\partial\eta} = sb(s)I_1(s\eta)$$

the boundary condition at $\eta = 1$ yields

$$sb(s)I_1(s) = -\frac{s^2\bar{A}(s)K_1(s)}{2\pi}\ ; \qquad b(s) = -\frac{s\bar{A}(s)K_1(s)}{2\pi I_1(s)}$$

Thus $\bar{\varphi}_2$ is given as follows:

$$\bar{\varphi}_2(s;\eta) = -\frac{s\bar{A}(s)K_1(s)I_o(s\eta)}{2\pi I_1(s)}\ ; \qquad \frac{\partial\bar{\varphi}_2}{\partial\eta} = -\frac{s^2\bar{A}(s)K_1(s)I_1(s\eta)}{2\pi I_1(s)} \tag{27}$$

Let the potential φ over the rear of the body be written as $\varphi(\xi,\eta) = \varphi_2(\xi,\eta) + \varphi_3(\xi,\eta)$ where $\bar{\varphi}_3$ is assumed to be given by the terms of the form $a_3(s)K_0(s\eta)$. In the previous case of the airfoil, the normal gradient of φ_2 at the airfoil surface was of the same order as the imposed boundary condition and it was necessary to take proper regard of φ_2 when equation (10a) was satisfied. In the present case the normal gradient of φ_2 is of higher order (at the body surface) in comparison with the contribution of φ_3, and the boundary condition is satisfied within the accuracy of the theory by the relation

$$a_3(s) = -\frac{1}{2\pi} s\bar{A}(s) , \qquad A(\xi) = \pi\beta_0 f^2(\xi) \tag{28}$$

The solution over the rear of the body is therefore

$$\bar{\varphi} = -\frac{s\bar{A}(s)K_1(s)I_0(s\eta)}{2\pi I_1(s)} - \frac{s\bar{A}(s)K_0(s\eta)}{2\pi} \tag{29}$$

In terms of the physical variables, the drag D of the configuration can be written as an integral of pressure over the surface of the body:

$$\frac{D}{q_0} = \int_0^l \left(\frac{p - p_0}{q_0}\right)_b 2\pi F(x) \frac{dF}{dx} dx \tag{30}$$

Since for slender bodies of revolution the pressure-velocity relation becomes

$$\frac{p - p_0}{q_0} = -\frac{2}{U_0}\frac{\partial\Phi}{\partial x} - \frac{1}{U_0^2}\left(\frac{\partial\Phi}{\partial y}\right)^2$$

it follows that the drag integral is, in terms of the dimensionless variables,

$$\frac{D}{q_0} = \beta_0 h^2 \int_0^{2/m} \left[-\frac{2}{\beta_0}\frac{\partial\varphi}{\partial\xi} - \left(\frac{\partial\varphi}{\partial\eta}\right)^2\right] \frac{dA}{d\xi} d\xi \tag{31}$$

where the Laplace transforms of φ are given by equation (26) for $0 \leq \xi \leq 2$ and by equation (29) for $2 \leq \xi \leq 2/m \leq 4$. Proceeding to the evaluation of the drag integral, we find from equation (26) that for small values of η

$$\left.\begin{aligned}\frac{\partial\bar{\varphi}}{\partial\xi} &= -\frac{1}{2\pi}\, s^2\bar{A}(s)K_0(s\eta) \approx \frac{s^2\bar{A}(s)}{2\pi}\left(\ln\frac{s\eta}{2} + \gamma_0\right)\\ \frac{\partial\bar{\varphi}}{\partial\eta} &= \frac{1}{2\pi}\, s^2\bar{A}(s)K_1(s\eta) \approx \frac{s\bar{A}(s)}{2\pi\eta}\end{aligned}\right\} \tag{32a}$$

where γ_0 is Euler's constant. From equation (29), similarly,

$$\left.\begin{aligned}\frac{\partial\bar{\varphi}}{\partial\xi} &= -\frac{s^2}{2\pi}\,\bar{A}(s)K_0(s\eta) - \frac{s^2}{2\pi}\,\bar{A}(s)\,\frac{K_1(s)}{I_1(s)}\,I_0(s\eta)\\ &\approx \frac{s^2}{2\pi}\,\bar{A}(s)\left(\ln\frac{s\eta}{2} + \gamma_0\right) - \frac{s^2}{2\pi}\,\bar{A}(s)\,\frac{K_1(s)}{I_1(s)}\\ \frac{\partial\bar{\varphi}}{\partial\eta} &= \frac{s^2}{2\pi}\,\bar{A}(s)K_1(s\eta) - \frac{s^2}{2\pi}\,\bar{A}(s)\,\frac{K_1(s)}{I_1(s)}\,I_0'(s\eta)\\ &\approx \frac{s\bar{A}(s)}{2\pi\eta}\end{aligned}\right\} \tag{32b}$$

From these expressions, the inverse transform can be written explicitly on the surface of the body in the following form:

for $0 \leq \xi \leq 2$

$$\left.\begin{aligned}\frac{\partial\varphi}{\partial\xi} &= \frac{1}{2\pi}\,A''(\xi)\ln\frac{\beta_0 f}{2} - \frac{1}{2\pi}\,\frac{\partial}{\partial\xi}\int_0^{\xi} A''(\xi_1)\ln(\xi - \xi_1)d\xi_1\\ \frac{\partial\varphi}{\partial\eta} &= \frac{1}{2\pi\beta_0 f(\xi)}\,\frac{dA}{d\xi} = \frac{df}{d\xi}\end{aligned}\right\} \tag{33a}$$

for $2 \leq \xi \leq 2/m \leq 4$

$$\left.\begin{aligned}\frac{\partial\varphi}{\partial\xi} &= \frac{1}{2\pi}\,A''(\xi)\ln\frac{\beta_0 f}{2} - \frac{1}{2\pi}\,\frac{\partial}{\partial\xi}\int_0^{\xi} A''(\xi_1)\ln(\xi - \xi_1)d\xi_1 - \frac{1}{2}\,A''(\xi - 2) - \\ &\qquad \frac{1}{2}\int_2^{\xi} A'''(\xi_1 - 2)\Omega(\xi - \xi_1)d\xi_1 - \frac{1}{2}\,A''(0)\Omega(\xi - 2)\\ \frac{\partial\varphi}{\partial\eta} &= \frac{df}{d\xi}\end{aligned}\right\} \tag{33b}$$

where $\Omega(\xi)$ is an influence function whose Laplace transform is

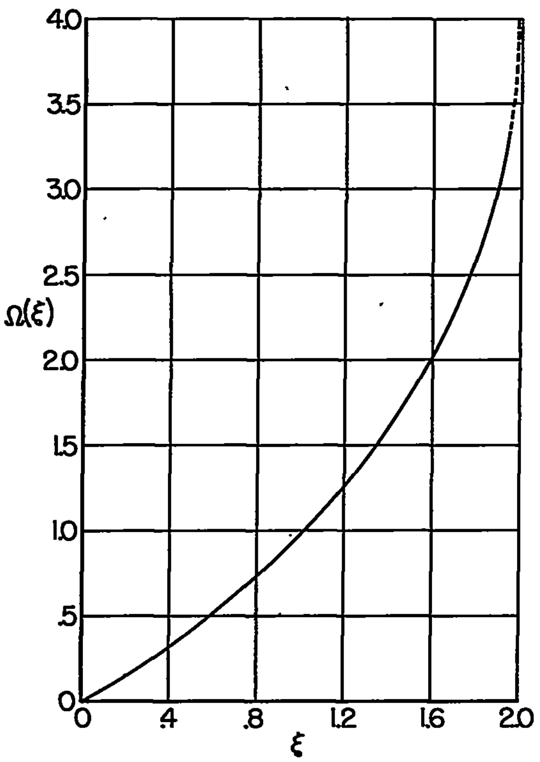
$$\bar{\Omega}(s) = \frac{1}{s}\left[\frac{e^{s}K_1(s)}{\pi e^{-s}I_1(s)} - 1\right] \tag{34}$$

and where the usual restrictions $A(0) = A'(0) = 0$ have now been placed upon $A(\xi)$.

The evaluation of the inverse transform of $\bar{\Omega}(s)$ is given in the appendix. It is shown that over the range of $0 \leq \xi \leq 2$ the relation

$$\Omega(\xi) = -\frac{1}{\pi}\ln\frac{2-\xi}{2} + T(\xi) \tag{35}$$

holds, where $T(\xi)$ is a power series that is convergent over the range. Sketch (i) shows $\Omega(\xi)$, and the accompanying table lists values of the function and its component parts for $0 \leq \xi \leq 2$.



Sketch (i)

ξ	$-\frac{1}{\pi}\ln\frac{2-\xi}{2}$	$T(\xi)$	$\Omega(\xi)$
0.20	0.033538	0.122638	0.156176
.40	.071029	.246073	.317102
.60	.113534	.403432	.516966
.80	.162601	.568452	.731053
1.00	.220636	.755683	.976319
1.20	.291665	.970676	1.262341
1.40	.383237	1.220196	1.603433
1.60	.512300	1.512421	2.024721
1.80	.732936	1.857148	2.590084
1.95	1.174207	2.157066	3.331273
2.00	∞	2.266000	∞

Equations (33) now must be substituted into equation (31). The resultant expression is the drag of the configuration. The linearity of the term $\partial\varphi/\partial\xi$ in the integrand, moreover, permits one to write drag in the form $D = D_{bb} + D_{bi}$ where D_{bb} is the drag of the body as it exists alone in its own induced flow field and D_{bi} is the drag of the body attributable to the induced effects of the shrouding tube.

Up to this point in the analysis no explicit use has been made of the assumption that the body closes and, in fact, straightforward evaluation of D_{bb} leads to the expression given by Ward (ref. 4) and Frankl and Karpovich (ref. 7) for open-ended bodies with finite slopes at the base. If, however, the body closes or has zero slope at the end, there results the simpler drag formula of von Kármán (ref. 8).

$$\begin{aligned}\frac{D_{bb}}{q_o} &= -\frac{h^2}{2\pi}\int_0^{2/m} A''(\xi)d\xi\int_0^{2/m} A''(\xi_1)\ln|\xi - \xi_1|d\xi_1 \\ &= \frac{h^2}{2\pi}\int_0^{2/m}\int_0^{2/m} A'(\xi)A''(\xi_1)\frac{d\xi\, d\xi_1}{\xi - \xi_1}\end{aligned} \tag{36}$$

The interference drag is then

$$\begin{aligned}\frac{D_{bi}}{q_o} &= \beta_o h^2\int_2^{2/m}\left[-\frac{2}{\beta_o}\frac{\partial\varphi_2}{\partial\xi} - \left(\frac{\partial\varphi_2}{\partial\eta}\right)^2\right]_b\frac{dA}{d\xi}d\xi \\ &= -h^2\int_2^{2/m} 2\frac{\partial\varphi_2}{\partial\xi}\frac{dA}{d\xi}d\xi\end{aligned}$$

where $\partial\varphi_2/\partial\xi$ is given by the last two terms in the right member of equation (33b). Substitution into the equation for the interference drag yields

$$\frac{D_{bi}}{q_o} = h^2\int_2^{2/m} A'(\xi)\left[A'''(\xi - 2) + A''(0)\Omega(\xi - 2) + \int_2^{\xi} A'''(\xi_1 - 2)\Omega(\xi - \xi_1)d\xi_1\right]d\xi \tag{37}$$

so that the formula for the total wave drag for a closed body of arbitrary shape becomes

$$\left.\begin{aligned}\frac{D}{q_o} &= h^2\int_2^{2/m} A'(\xi)\left[A''(\xi - 2) + A''(0)\Omega(\xi - 2) + \int_2^{\xi} A'''(\xi_1 - 2)\Omega(\xi - \xi_1)d\xi_1\right]d\xi + \\ &\quad \frac{h^2}{2\pi}\int_0^{2/m} A'(\xi)d\xi\int_0^{2/m}\frac{A''(\xi_1)}{\xi - \xi_1}d\xi_1, \qquad 1/2 \le m \le 1 \\ \frac{D}{q_o} &= \frac{h^2}{2\pi}\int_0^{2/m} A'(\xi)d\xi\int_0^{2/m}\frac{A''(\xi_1)}{\xi - \xi_1}d\xi_1, \qquad m > 1\end{aligned}\right\} \tag{38a}$$

In terms of the physical variables, using equation (25), one may write the drag in the final form

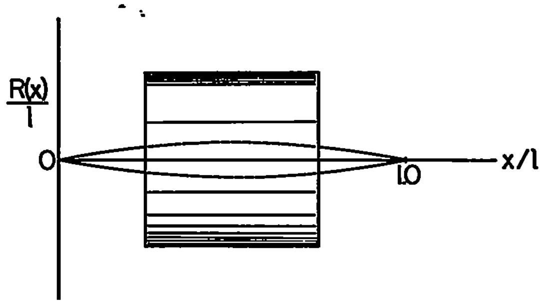
$$\left.\begin{aligned}\frac{D}{q_o} &= \int_{ml}^{l} S'(x)\left[S''(x - ml) + S'''(0)\Omega\left(\frac{2x}{ml} - 2\right) + \int_0^{x-ml} S'''(x - ml - x_1)\Omega\left(\frac{2x_1}{ml}\right)dx_1\right]dx + \\ &\quad \frac{1}{2\pi}\int_0^l S'(x)dx\int_0^l \frac{S''(x_1)}{x - x_1}\,dx_1\,, \qquad 1/2 \leq m \leq 1 \\ \frac{D}{q_o} &= \frac{1}{2\pi}\int_0^l S'(x)dx\int_0^l \frac{S''(x_1)}{x - x_1}\,dx_1\,, \qquad m > 1\end{aligned}\right\} \tag{38b}$$

where $m = 2\beta_o h/l$. If the shrouding tube of the configuration is designed for a fixed radius h, formulas (38) give the drag for a shrouded body of revolution of arbitrary shape for Mach numbers $M_o \geq \sqrt{1 + l^2/16h^2}$. When $m = 1/2$ the waves from the forward portion of the body are reflected onto the rearward as in sketch (a). For $m \geq 1$ there is no effect of the tube and the formula for the drag is the same as that of the body without shroud.

As an example, consider the body of revolution whose generating curve is defined by

$$R(x) = \frac{2t_o}{l^2}\,x(l - x)\,, \qquad 0 \leq x \leq l \tag{39a}$$

where t_o is the maximum thickness (see sketch (j)). Here



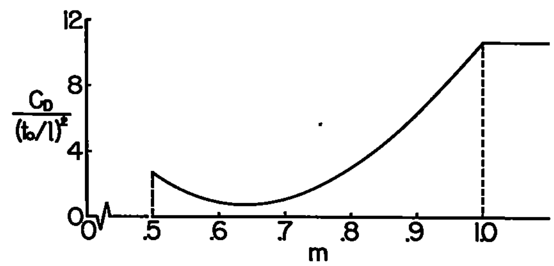
Sketch (j)

$$\left.\begin{aligned}S(x) &= \frac{4\pi t_o^2}{l^4}\,x^2(l - x)^2 \\ S'(x) &= \frac{8\pi t_o^2}{l^4}\,x(l - x)(l - 2x) \\ S''(x) &= \frac{8\pi t_o^2}{l^4}\,(l^2 - 6xl + 6x^2) \\ S'''(x) &= \frac{48\pi t_o^2}{l^4}\,(2x - l)\end{aligned}\right\} \tag{39b}$$

with frontal area $S_F = \pi t_o^2/4$. For $m > 1$ the second formula in equations (38b) yields

$$C_D = \frac{D}{q_o S_f} = \frac{32}{3}\left(\frac{t_o}{l}\right)^2 \qquad (40)$$

Integration of the first relation in formulas (38b) for $1/2 \leq m \leq 1$ is carried out numerically and the results are shown in sketch (k). For a fixed value of the ratio h/l, this sketch gives the drag of the configuration as a function of β_o for $\beta_o \geq l/4h$.



Sketch (k)

Shrouded Bodies Having Zero Wave Drag

In general, the value found from the drag formula (38) for a shrouded body will be greater than zero. It is possible to obtain from this formula, however, an integral equation whose solution will yield a class of body shapes for which the wave drag is zero.[1]

Derivation of integral equation.- In the particular application to be considered here, we take $\beta_o h = l/4$ $(m = 1/2)$ and assume that $A''(0)=0$. A further assumption to be made is that the body has fore-and-aft symmetry. This implies that $A(\xi) = A(4 - \xi)$ and, since no discontinuities are allowed in the slope of the meridian section, $A'(2) = 0$. Under these conditions the first relation in equation (38a) becomes

$$\frac{D}{q_o} = h^2 \int_2^4 A'(\xi)\left[A''(\xi - 2) + \int_2^{\xi} A'''(\xi_1 - 2)\Omega(\xi - \xi_1)d\xi_1 + \frac{1}{\pi}\int_0^4 \frac{A''(\xi_1)}{\xi - \xi_1}\, d\xi_1\right]d\xi \qquad (41)$$

A sufficient condition for the drag to vanish is that the expression within the brackets be zero, resulting in the relation

$$A''(\xi - 2) + \int_2^{\xi} A'''(\xi_1 - 2)\Omega(\xi - \xi_1)d\xi_1 + \frac{1}{\pi}\int_0^4 \frac{A''(\xi_1)}{\xi - \xi_1}\, d\xi_1 = 0\ , \qquad 2 \leq \xi \leq 4 \qquad (42a)$$

The solution of this integral equation for the function $A(\xi)$ is required in order to find the area distribution $S(x) = \beta_o h^2 A(\xi)$ of the body whose drag is canceled by the shrouding tube.

[1]While this part of the analysis was being carried out, Graham, Beane, and Licher (ref. 9) published a paper treating essentially the same problem from a different point of view.

Solution of integral equation.- Equation (42a) can also be written

$$E(\xi) = A''(\xi) + \int_0^{\xi} A''(\xi_1)\psi(\xi - \xi_1)d\xi_1 , \qquad 0 \leq \xi \leq 2 \tag{42b}$$

where

$$\left.\begin{aligned} E(\xi) &= -\frac{1}{\pi}\int_0^2\left(\frac{1}{\xi - 2 - \xi_1} + \frac{1}{\xi - 2 + \xi_1}\right)A''(\xi_1)d\xi_1 \\ \psi(\xi) &= \frac{d}{d\xi}\,\Omega(\xi) \end{aligned}\right\} \tag{43}$$

In dealing with this integral equation it is advantageous to apply the calculus of Laplace transform because the equation contains the convolution of A'' and ψ.

Taking the Laplace transform of both sides of relation (42b), one obtains

$$\bar{E}(s) = s^2\bar{A}(s)[1 + s\bar{\Omega}(s)] \tag{44a}$$

or

$$\bar{E}(s) = s^2\bar{A}(s)\,\frac{e^{s}K_1(s)}{\pi e^{-s}I_1(s)} \tag{44b}$$

For a homogeneous equation of the type (42b) it is usually convenient to assume that the unknown can be expressed in the form

$$A''(\xi) = \int_0^{\xi} \gamma(\xi_2)g(\xi - \xi_2)d\xi_2 \tag{45}$$

where the functions g and γ are to be determined. Equation (44b) then yields

$$\bar{E}(s) = \bar{\gamma}(s)\bar{g}(s)\,\frac{e^{s}K_1(s)}{\pi e^{-s}I_1(s)} \tag{46}$$

Now the quantity in the denominator of this relation can be canceled by simply assuming

$$\bar{\gamma}(s) = \pi e^{-s}I_1(s) \tag{47a}$$

or

$$\left.\begin{aligned}\gamma(\xi_2) &= \frac{1-\xi_2}{\sqrt{\xi_2(2-\xi_2)}}\,, \qquad 0 \leq \xi_2 \leq 2\\ &= 0\,, \qquad\qquad\qquad\quad \xi_2 > 2\end{aligned}\right\} \tag{47b}$$

so that equation (46) reduces to

$$\bar{E}(s) = \bar{g}(s)e^{s}K_1(s) \tag{48a}$$

or

$$E(\xi) = \int_0^{\xi} \frac{1+\xi_1}{\sqrt{\xi_1(2+\xi_1)}}\, g(\xi-\xi_1)d\xi_1 \tag{48b}$$

From equations (45), (47b), (48b), and the first relation in equations (43), there results the following integral equation for determining the function g:

$$\int_0^{\xi} \frac{1+\xi_1}{\sqrt{\xi_1(2+\xi_1)}}\, g(\xi-\xi_1)d\xi_1$$

$$= -\frac{1}{\pi}\int_0^2 \left[\frac{1}{\xi-2-\xi_1} + \frac{1}{\xi-2+\xi_1}\right]d\xi_1 \int_0^{\xi_1} \frac{1-\xi_2}{\sqrt{\xi_2(2-\xi_2)}}\, g(\xi_1-\xi_2)d\xi_2 \tag{49}$$

After integration and some manipulation, relation (49) reduces to the homogeneous equation of the first kind

$$\int_0^2 G_1(\xi_1)k(\xi,\xi_1)d\xi_1 = 0\,, \qquad 0 \leq \xi \leq 2 \tag{50}$$

with unknown

$$G_1(\xi_1) = g(\xi_1) + g(2-\xi_1) \tag{51}$$

and kernel

$$k(\xi,\xi_1) = \frac{1+\xi-\xi_1}{\sqrt{(\xi-\xi_1)(2+\xi-\xi_1)}}\tan^{-1}\sqrt{\frac{(2-\xi_1)(\xi-\xi_1)}{(2+\xi-\xi_1)\xi_1}} \tag{52}$$

This expression obviously has at least the trivial solution $G_1(\xi_1) \equiv 0$, or

$$g(\xi_1) = -\,g(2-\xi_1) \ , \qquad 0 \le \xi_1 \le 2 \tag{53}$$

requiring simply that the function g in equation (45) be odd about $\xi_1 = 1$. For a function that satisfies this condition, it follows that a formal solution of the integral equation (42) can be written as

$$A''(\xi) = \int_0^{\xi} \frac{1-\xi_1}{\sqrt{\xi_1(2-\xi_1)}}\, g(\xi-\xi_1)d\xi_1 \ , \qquad 0 \le \xi \le 2 \tag{54a}$$

or

$$A'(\xi) = \int_0^{\xi} \sqrt{\xi_1(2-\xi_1)}\, g(\xi-\xi_1)d\xi_1 \tag{54b}$$

In order to avoid obtaining bodies with negative areas, however, the function g satisfying relation (53) must be chosen so that

$$A(\xi) = \frac{1}{2}\int_0^{\xi} \left[(\xi_1-1)\sqrt{\xi_1(2-\xi_1)} + \cos^{-1}(1-\xi_1)\right] g(\xi-\xi_1)d\xi_1 \tag{54c}$$

is greater than or equal to zero. These formulas determine the shape of the shrouded body for the range $0 \le \xi \le 2$, while the rear half of the body is obtained by reflection about $\xi = 2$. There is of course an infinitude of functions for which the relation (53) holds, so that equations (54) actually furnish an infinite class of body shapes whose drag is canceled by a shrouding cylindrical shell. It will be shown later that this function g can be related to the pressure coefficient on the shroud.

The quantities defined by formulas (54) are continuous if g is piecewise continuous. It is apparent from these equations that the restriction $A''(0) = A'(0) = A(0) = 0$ is met, and it can be seen by use of equation (53) that the requirement $A'(2) = 0$ is also satisfied. Moreover, it can be shown that if A' is required to vanish at $\xi = 2$, and if g is restricted to be piecewise continuous, the solution $G_1(\xi_1) \equiv 0$ used for equation (50) is the only one admissible here.

A simple function that satisfies the functional relation (53) is

$$\left.\begin{aligned} g(\xi_1) &= \lambda_2\xi_1 \ , && 0 \le \xi_1 \le 1 \\ &= \lambda_2(\xi_1-2) \ , && 1 \le \xi_1 \le 2 \end{aligned}\right\} \tag{55}$$

Inserting this expression into the formula (54c) yields, for $1 \le \xi \le 2$,

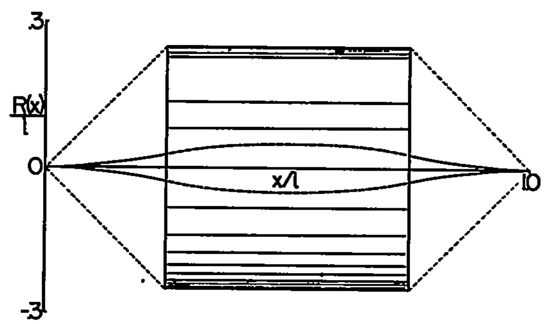
$$A(\xi) = \frac{\lambda_2}{48}\Big[3(4\xi^2 - 8\xi + 5)\cos^{-1}(1 - \xi) + (\xi - 1)(2\xi^2 - 4\xi + 15)\sqrt{\xi(2 - \xi)} +$$

$$48(2 - \xi)\cos^{-1}(2 - \xi) - 16(\xi^2 - 4\xi + 6)\sqrt{(\xi - 1)(3 - \xi)}\Big] \tag{56a}$$

and, for $0 \leq \xi \leq 1$,

$$A(\xi) = \frac{\lambda_2}{48}\Big[3(4\xi^2 - 8\xi + 5)\cos^{-1}(1 - \xi) + (\xi - 1)(2\xi^2 - 4\xi + 15)\sqrt{\xi(2 - \xi)}\Big] \tag{56b}$$

A plot of the coordinates of the body is shown in sketch (l) for the special case when the maximum thickness ratio is 0.1 and the Mach number is $\sqrt{2}$.



Sketch (l)

A simpler example is given by taking

$$g(\xi_1) = \lambda(1 - \xi_1)\ , \qquad 0 \leq \xi_1 \leq 2 \tag{57}$$

Substitution into formula (54c) leads, for $0 \leq \xi \leq 2$, to

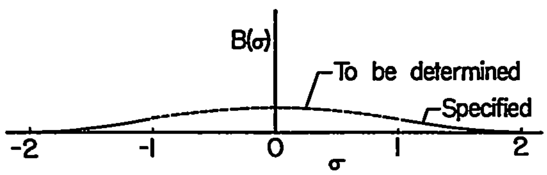
$$A(\xi) = \frac{\lambda}{48}\Big[3(-4\xi^2 + 16\xi - 13)\cos^{-1}(1 - \xi) + (-2\xi^3 + 14\xi^2 - 35\xi + 39)\sqrt{\xi(2 - \xi)}\Big] \tag{58}$$

a result which is the same as that obtained by Graham, Beane, and Licher (ref. 9). If the quantity λ in equation (58) is chosen so that the body considered in reference 9 has the same volume as that of the previous example, the former body has a maximum cross-sectional area 0.94 times as great.

<u>Determination of body shape when a portion of it is specified</u>.- For a function satisfying relation (53), it is seen that formulas (54) yield a class of bodies having zero drag. Instead of prescribing the function g and then calculating A, however, it is possible to find the entire shape of the body when only a portion of it is given. This is accomplished by starting with equation (42a).

Equation (42a) can be put in an improved form by translating the axes so that the origin is at the body midpoint. To this end, set

$$\left.\begin{aligned} \xi = \sigma + 2 \;, \qquad \xi_1 = \sigma_1 + 2 \\ A(\xi) = A(4 - \xi) = A(2 + \sigma) = A(2 - \sigma) = B(\sigma) = B(-\sigma) \end{aligned}\right\} \tag{59}$$



Sketch (m)

As shown in sketch (m) equation (42a) can then be interpreted as an integral equation in which $B(\sigma)$ is specified in the range $-2 \leq \sigma \leq -1$ or $1 \leq \sigma \leq 2$, and it is required to find B in the range $0 \leq \sigma \leq 1$ by inverting the expression

$$\mu(\sigma) = -\frac{2\sigma}{\pi}\int_0^1 \frac{B''(\sigma_1)}{\sigma^2 - \sigma_1^2}\, d\sigma_1 \tag{60}$$

where $\mu(\sigma)$ is known and is given by

$$\mu(\sigma) = B''(2 - \sigma) + \int_0^\sigma B''(2 - \sigma_1)\psi(\sigma - \sigma_1)d\sigma_1 + \frac{2\sigma}{\pi}\int_0^1 \frac{B''(\sigma_1)}{\sigma^2 - \sigma_1^2}\, d\sigma_1 \tag{61}$$

with $\psi(\sigma) = d\Omega/d\sigma$. A final change of variables $(\sigma^2 = \tau)$ reduces equation (60) to the form

$$\frac{\mu(\sqrt{\tau})}{4\sqrt{\tau}} = -\frac{1}{2\pi}\int_0^1 \frac{(d/d\tau_1)[2\sqrt{\tau_1}(d/d\tau_1)B(\sqrt{\tau_1})]}{\tau - \tau_1}\, d\tau_1 \tag{62}$$

The singular integral equation (62) is a familiar one in aerodynamics. Since

$$\int_0^1 \frac{d}{d\tau_1}\left[2\sqrt{\tau_1}\,\frac{d}{d\tau_1}\, B(\sqrt{\tau_1})\right] d\tau_1 = B'(1)$$

its inversion is

$$\frac{d}{d\tau}\left[\sqrt{\tau}\,\frac{d}{d\tau}\, B(\sqrt{\tau})\right] = \frac{1}{4\pi\sqrt{1 - \tau}}\left[2B'(1) + \int_0^1 \frac{\mu(\sqrt{\tau_1})\sqrt{1 - \tau_1}}{\tau - \tau_1}\, d\tau_1\right]$$

or

$$B''(\sigma) = \frac{2}{\pi\sqrt{1-\sigma^2}}\left[B'(1) + \int_0^1 \frac{\tau_2\mu(\tau_2)\sqrt{1-\tau_2^2}}{\sigma^2 - \tau_2^2}\,d\tau_2\right] \tag{63}$$

Equation (63) can be integrated with respect to σ; by interchanging the order of integration in the last term and imposing the condition $B'(0) = 0$, we find

$$B'(\sigma) = \frac{2B'(1)}{\pi}\sin^{-1}\sigma + \frac{1}{\pi}\int_0^1 \mu(\tau_2)\ln\left|\frac{\tau_2\sqrt{1-\sigma^2} - \sigma\sqrt{1-\tau_2^2}}{\tau_2\sqrt{1-\sigma^2} + \sigma\sqrt{1-\tau_2^2}}\right|d\tau_2 \tag{64}$$

Integrating once again with respect to σ finally yields the expression

$$B(\sigma) = B(1) + \frac{2B'(1)}{\pi}\left[\sqrt{1-\sigma^2} + \sigma\sin^{-1}\sigma - \frac{\pi}{2}\right] +$$

$$\frac{1}{\pi}\int_0^1 \mu(\tau_2)\left[\sigma\ln\left|\frac{\tau_2\sqrt{1-\sigma^2} - \sigma\sqrt{1-\tau_2^2}}{\tau_2\sqrt{1-\sigma^2} + \sigma\sqrt{1-\tau_2^2}}\right| - \tau_2\ln\left|\frac{\sqrt{1-\sigma^2} - \sqrt{1-\tau_2^2}}{\sqrt{1-\sigma^2} + \sqrt{1-\tau_2^2}}\right|\right]d\tau_2 \tag{65}$$

for the value of B in the interval $0 \leq \sigma \leq 1$ when it is prescribed for $1 \leq \sigma \leq 2$.

As an example, specify $B''(\sigma)$ to be

$$B''(\sigma) = \lambda\sqrt{\sigma(2-\sigma)} \tag{66}$$

in the interval $1 \leq \sigma \leq 2$ and find its value for the range $0 \leq \sigma \leq 1$. It can be shown that

$$\int_0^\sigma B''(2-\sigma_1)\psi(\sigma-\sigma_1)d\sigma_1 = \lambda\int_0^\sigma \sqrt{\sigma_1(2-\sigma_1)}\psi(\sigma-\sigma_1)d\sigma_1$$

$$= \lambda\left[\sqrt{\sigma(2+\sigma)} - \sqrt{\sigma(2-\sigma)}\right] \tag{67}$$

so that, from equation (61),

$$\mu(\sigma) = \lambda\sqrt{\sigma(2+\sigma)} + \frac{2\sigma\lambda}{\pi}\int_0^2 \frac{\sqrt{\sigma_1(2-\sigma_1)}}{\sigma^2 - \sigma_1^2}\,d\sigma_1 \tag{68}$$

Substitution of this quantity into the inversion formula (63) leads, after integration, to the result

$$B''(\sigma) = \frac{2\lambda}{\pi\sqrt{1-\sigma^2}}\left[\int_0^1 \frac{\sigma_1\sqrt{\sigma_1(2+\sigma_1)(1-\sigma_1^2)}}{\sigma^2-\sigma_1^2}\,d\sigma_1 + \int_1^2 \frac{\tau_2\sqrt{\tau_2(2-\tau_2)(\tau_2^2-1)}}{\sigma^2-\tau_2^2}\,d\tau_2\right] \tag{69}$$

or finally

$$B''(\sigma) = \lambda\sqrt{\sigma(2-\sigma)} - 2\lambda\sqrt{1-\sigma^2}\,, \qquad 0 \le \sigma \le 1 \tag{70a}$$

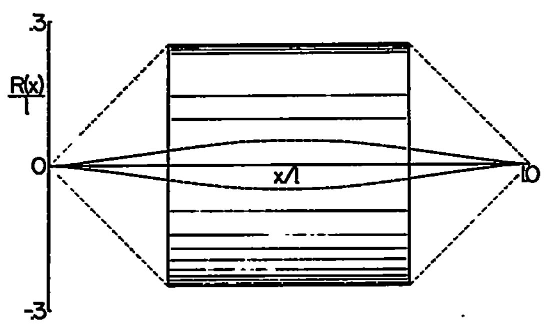
Integration of this then gives

$$B'(\sigma) = \frac{\lambda}{2}\left[(\sigma-1)\sqrt{\sigma(2-\sigma)} - \cos^{-1}(\sigma-1) - 2\sigma\sqrt{1-\sigma^2} + 2\cos^{-1}\sigma\right] \tag{70b}$$

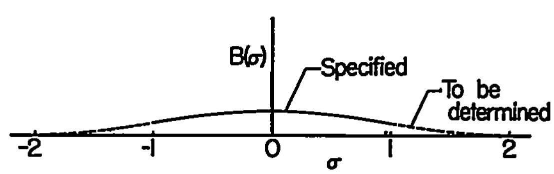
or

$$B(\sigma) = \frac{\lambda}{6}\left[(\sigma^2-2\sigma+3)\sqrt{\sigma(2-\sigma)} - 2(\sigma^2+2)\sqrt{1-\sigma^2} + 6\sigma\cos^{-1}\sigma + 3(1-\sigma)\cos^{-1}(\sigma-1)\right], \qquad 0 \le \sigma \le 1 \tag{70c}$$

Sketch (n) shows a plot of the coordinates of the body when the maximum thickness ratio is 0.1 and $M_0 = \sqrt{2}$.



Sketch (n)



Sketch (o)

If the function $B(\sigma)$ is specified in the range $-1 \le \sigma \le 0$ or $0 \le \sigma \le 1$, equation (42a) may also be employed to find $B(\sigma)$ for $-2 \le \sigma \le -1$ or $1 \le \sigma \le 2$. (See sketch (o).) In this case we may write the relation (42a) in the form

$$B''(2-\sigma) - \int_1^\sigma B'''(2-\sigma)\Omega(\sigma-\sigma_1)d\sigma_1 + \frac{2\sigma}{\pi}\int_0^1 \frac{B''(\sigma_1)}{\sigma^2-\sigma_1^2}\,d\sigma_1 -$$

$$\int_0^1 B'''(2-\sigma_1)\Omega(\sigma-\sigma_1)d\sigma_1 = -\frac{2\sigma}{\pi}\int_1^2 \frac{B''(\sigma_1)}{\sigma^2-\sigma_1^2}\,d\sigma_1 \; ; \qquad 1 \leq \sigma \leq 2 \tag{71}$$

where the last term on the left, as well as the term on the right is unknown. Applying the inversion formula for the airfoil equation under the condition that $B''(2) = 0$, yields the equation

$$G(\sigma) = B''(\sigma) + \int_1^2 H(\sigma,\sigma_2)B''(\sigma_2)d\sigma_2 \tag{72}$$

where the known functions G and H are given by

$$\left.\begin{aligned}
G(\sigma) &= \frac{2\sigma}{\pi}\sqrt{\frac{4-\sigma^2}{\sigma^2-1}}\int_1^2 \frac{h_1(\tau_2)}{\sigma^2-\tau_2^2}\sqrt{\frac{\tau_2^2-1}{4-\tau_2^2}}\,d\tau_2 \\
h_1(\tau_2) &= B''(2-\tau_2) + \int_{2-\tau_2}^1 B''(\sigma_1)\psi(\tau_2-2+\sigma_1)d\sigma_1 + \frac{2\tau_2}{\pi}\int_0^1 \frac{B''(\sigma_1)}{\tau_2^2-\sigma_1^2}\,d\sigma_1 \\
H(\sigma,\sigma_2) &= \frac{2\sigma}{\pi}\sqrt{\frac{4-\sigma^2}{\sigma^2-1}}\int_1^2 \frac{d\tau_2}{\sigma^2-\tau_2^2}\sqrt{\frac{\tau_2^2-1}{4-\tau_2^2}}\,\psi(\tau_2-2+\sigma_2)
\end{aligned}\right\} \tag{73}$$

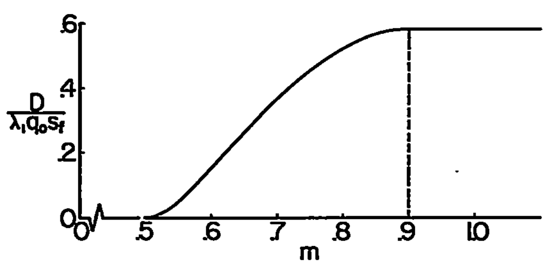
Relation (72) is a Fredholm integral equation of the second kind and can be treated by well-known methods (see, e.g., ref. 10), although an inversion formula for the equation cannot, in general, be written in closed form as was done (eq. (62)) for the previous case.

<u>Drag for off-design condition</u>.- If the body geometry satisfies the integral equation (42a), the drag will be zero only at the design condition $\beta_0 = l/4h$ $(m = 1/2)$. The value of the drag of such a body for $1/2 < m < 1$, however, can be calculated by means of equation (38a) and will lie between zero and that for the body alone $(m \geq 1)$. Substitution from equation (42a) into formula (38b) gives the equation

$$\left.\begin{aligned}
\frac{D}{q_o} &= \int_{ml}^{l} S'(x)\left[S''(x-ml)+\int_{0}^{x-ml} S'''(x-ml-x_1)\Omega\left(\frac{2x_1}{ml}\right)dx_1\right]dx - \\
&\quad \int_{l/2}^{l} S'(x)\left[S''\left(x-\frac{l}{2}\right)+\int_{0}^{x-(l/2)} S'''\left(x-\frac{l}{2}-x_1\right)\Omega\left(\frac{4x_1}{l}\right)dx_1\right]dx\ ; \qquad \frac{1}{2}\le m\le 1 \\
\frac{D}{q_o} &= -\int_{l/2}^{l} S'(x)\left[S''\left(x-\frac{l}{2}\right)+\int_{0}^{x-(l/2)} S'''\left(x-\frac{l}{2}-x_1\right)\Omega\left(\frac{4x_1}{l}\right)dx_1\right]dx\ ; \qquad m>1
\end{aligned}\right\} \tag{74}$$

Consider, as an example, the shrouded body whose geometry is defined by equation (58), that is,

$$\left.\begin{aligned}
S''(x) &= \frac{\lambda_1}{2}\left[4\left(3-4\frac{x}{l}\right)\sqrt{\frac{x}{l}\left(\frac{1}{2}-\frac{x}{l}\right)}-\cos^{-1}\left(1-4\frac{x}{l}\right)\right], \qquad 0\le x\le l/2 \\
S''(x) &= S''(l-x)\ , \qquad l/2\le x\le l
\end{aligned}\right\}$$



Sketch (p)

For $m = 1/2$ the configuration has zero drag, since A" or S" is a solution of the integral equation (42a). Integration of equation (74) is carried out for this particular case for $1/2 < m$ and the results are shown in sketch (p). Calculations for $m < 1/2$ become exceedingly involved and cannot be obtained from the analysis, although the same general method is applicable.

Pressure Distribution

Pressure coefficient of shrouded body of arbitrary shape.- It is of interest to determine the pressure distribution of a body of revolution shrouded by a cylindrical shell. From equations (33), the pressure coefficient

$$C_p = \frac{p-p_o}{q_o} = -\frac{2}{U_o}\frac{\partial\Phi}{\partial x}-\frac{1}{U_o^2}\left(\frac{\partial\Phi}{\partial y}\right)^2 = -\frac{2}{\beta_o}\frac{\partial\varphi}{\partial\xi}-\left(\frac{\partial\varphi}{\partial\eta}\right)^2$$

on the same body in the range $0 \leq \xi \leq 2$ is given by

$$C_p = -\frac{A''(\xi)}{\pi\beta_o} \ln \frac{\beta_o f}{2} + \frac{1}{\pi\beta_o} \frac{\partial}{\partial\xi} \int_0^{\xi} A''(\xi_1) \ln(\xi - \xi_1) d\xi_1 - \left(\frac{df}{d\xi}\right)^2 \tag{75a}$$

and, for $2 \leq \xi \leq 2/m \leq 4$, by

$$C_p = -\frac{A''(\xi)}{\pi\beta_o} \ln \frac{\beta_o f}{2} + \frac{1}{\pi\beta_o} \frac{\partial}{\partial\xi} \int_0^{\xi} A''(\xi_1) \ln(\xi - \xi_1) d\xi_1 - \left(\frac{df}{d\xi}\right)^2 + \frac{1}{\beta_o} A''(0)\Omega(\xi - 2) +$$

$$\frac{1}{\beta_o} A''(\xi - 2) + \frac{1}{\beta_o} \int_2^{\xi} A'''(\xi_1 - 2)\Omega(\xi - \xi_1) d\xi_1 \tag{75b}$$

In terms of the physical variables, using equation (25), we write these formulas in the final form

$$C_p = -\frac{S''(x)}{\pi} \ln \frac{\beta_o R}{2} - \left(\frac{dR}{dx}\right)^2 + \frac{1}{\pi} \frac{\partial}{\partial x} \int_0^x S'''(x_1) \ln(x - x_1) dx_1 \; ; \qquad 0 \leq x \leq ml \tag{76a}$$

$$C_p = -\frac{S''(x)}{\pi} \ln \frac{\beta_o R}{2} - \left(\frac{dR}{dx}\right)^2 + \frac{1}{\pi} \frac{\partial}{\partial x} \int_0^x S'''(x_1) \ln(x - x_1) dx_1 + S''(0)\Omega\left(\frac{2x}{ml} - 2\right) +$$

$$S''(x - ml) + \int_0^{x-ml} S'''(x - ml - x_1)\Omega\left(\frac{2x_1}{ml}\right) dx_1 \; ; \qquad ml \leq x \leq l \tag{76b}$$

In the interval $0 \leq x \leq ml$ the pressure coefficient is the same as that of the body without shroud, while for $ml \leq x \leq l$ it is affected by the last three terms in formula (76b). One sees from equations (76a) and (76b) that at $x = ml$ the coefficient has a jump equal to $S''(0)$, which means that the pressure is not continuous unless the body has a cusped nose.

For the particular example of the body whose generating curve is the parabolic arc

$$R(x) = \frac{2t_o}{l^2} x(l - x) \; , \qquad 0 \leq x \leq l \tag{77}$$

one obtains for the forward portion of the body

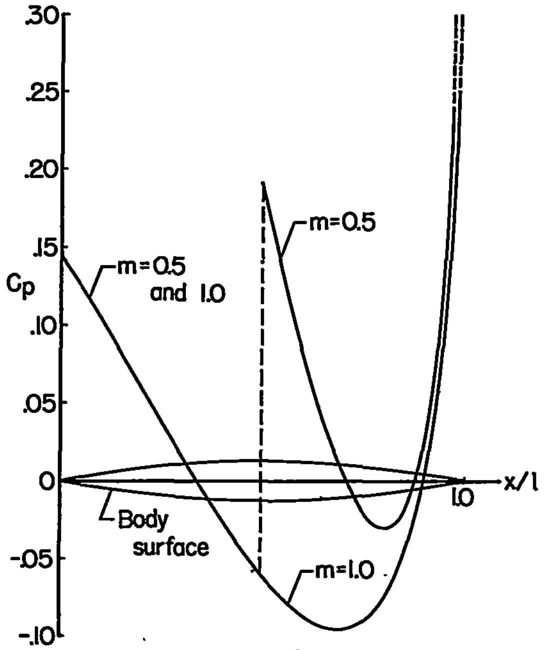
$$C_p = 4\left(\frac{t_o}{l}\right)^2\left[-1+16\frac{x}{l}-22\frac{x^2}{l^2}-2\left(1-6\frac{x}{l}+6\frac{x^2}{l^2}\right)\ln\frac{\beta_o t_o}{l}\left(1-\frac{x}{l}\right)\right]; \quad 0 \leq x \leq ml \tag{78a}$$

and for the rearward portion

$$C_p = 4\left(\frac{t_o}{l}\right)^2\left[-1+16\frac{x}{l}-22\frac{x^2}{l^2}+2\pi\left(1+6m+6m^2+6\frac{x^2}{l^2}-6\frac{x}{l}-12\frac{x}{l}m\right)-\right.$$

$$2\left(1-6\frac{x}{l}+6\frac{x^2}{l^2}\right)\ln\frac{\beta_o t_o}{l}\left(1-\frac{x}{l}\right)+2\pi\Omega\left(\frac{2x}{ml}-2\right)+$$

$$\left.12\pi\int_0^{(x/l)-m}\left(2\frac{x}{l}-2m-1-2z\right)\Omega\left(\frac{2}{m}z\right)dz\right]; \quad ml \leq x \leq l \tag{78b}$$



Sketch (q)

A sketch of the body of revolution having maximum thickness ratio equal to 0.1 and a plot of the corresponding pressure distribution at Mach number $\sqrt{2}$ are shown in sketch (q) for $m = 1/2$, and also for $m \geq 1$ (when the body is without shroud). The drag of the shrouded body is shown in sketch (k).

Pressure coefficient of zero-drag body.- When the body geometry is such that relation (42a) for zero drag holds, formula (75b) for the pressure coefficient on the rearward portion of the body (i.e., in the interval $2 \leq \xi \leq 4$) reduces to

$$C_p = -\frac{A''(\xi)}{\pi\beta_o}\ln\frac{\beta_o f}{2}-\frac{1}{\pi\beta_o}\frac{\partial}{\partial\xi}\int_\xi^4 A''(\xi_1)\ln(\xi_1-\xi)d\xi_1-\left(\frac{df}{d\xi}\right)^2$$

so that for $0 \leq x \leq l/2$

$$C_p = -\frac{S''(x)}{\pi}\ln\frac{\beta_o R}{2}-\left(\frac{dR}{dx}\right)^2+\frac{1}{\pi}\frac{\partial}{\partial x}\int_0^x S''(x_1)\ln(x-x_1)dx_1 \tag{79a}$$

and for $l/2 \leq x \leq l$

$$C_p = -\frac{S''(x)}{\pi} \ln \frac{\beta_o R}{2} - \left(\frac{dR}{dx}\right)^2 - \frac{1}{\pi} \frac{\partial}{\partial x} \int_x^l S''(x_1) \ln(x_1 - x) dx_1 \qquad (79b)$$

Since the body here is symmetrical fore and aft, that is, since $S''(x) = S''(l - x)$, it can be seen from equations (79a) and (79b) that the pressure distribution on the body also has fore-and-aft symmetry.

It remains to relate the pressure coefficient on the shroud to the function g in formulas (54). From the first relation in equation (32b)

$$\frac{\partial \bar{\varphi}}{\partial \xi} = -\frac{s^2 \bar{A}(s)}{2\pi I_1(s)} \left[I_1(s) K_o(s\eta) + K_1(s) I_o(s\eta)\right]$$

so that on the shroud (i.e., at $\eta = 1$)

$$\bar{C}_p = -\frac{2}{\beta_o} \frac{\partial \bar{\varphi}}{\partial \xi} = \frac{s^2 \bar{A}(s)}{\beta_o \pi I_1(s)} \left[I_1(s) K_o(s) + K_1(s) I_o(s)\right]$$

$$= \frac{s\bar{A}(s)}{\beta_o \pi I_1(s)} \qquad (80)$$

Taking the transform of equation (54a), one obtains

$$s^2 \bar{A}(s) = \pi e^{-s} I_1(s) \bar{g}(s)$$

or

$$\bar{g}(s) = \frac{s^2 \bar{A}(s)}{\pi e^{-s} I_1(s)} \qquad (81)$$

Thus from equations (80) and (81)

$$s\bar{C}_p = \frac{e^{-s} \bar{g}(s)}{\beta_o} \qquad (82)$$

so that the pressure coefficient on the shroud is related to g by

$$g(\xi) = \beta_o C_p'(1 + \xi) \qquad (83a)$$

It follows from this equation and relation (53) that

$$\left.\begin{aligned} \beta_o C_p &= \int_0^{\xi-1} g(\xi_1)d\xi_1 \,, \qquad 1 \leq \xi \leq 2 \\ \beta_o C_p &= \int_0^{3-\xi} g(\xi_1)d\xi_1 \,, \qquad 2 \leq \xi \leq 3 \end{aligned}\right\} \tag{83b}$$

or

$$C_p(1+\xi) = C_p(3-\xi) \tag{83c}$$

that is, that the pressure distribution on the shroud has fore-and-aft symmetry about the body midpoint $\xi = 2$.

Ames Aeronautical Laboratory
National Advisory Committee for Aeronautics
Moffett Field, Calif, Apr. 6, 1956

APPENDIX A

EVALUATION OF AN INVERSE TRANSFORM

The Laplace transform of the influence function[1] Ω was expressed in equation (34) by

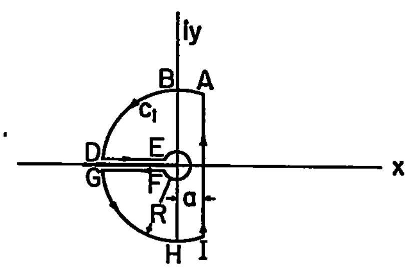
$$\bar{\Omega}(s) = \frac{1}{s}\left[\frac{e^{s}K_1(s)}{\pi e^{-s}I_1(s)} - 1\right] \tag{A1}$$

In this section, representations of the inverse transform

$$\Omega(\xi) = \lim_{\omega \to \infty} \frac{1}{2\pi i}\int_{a-i\omega}^{a+i\omega} e^{z\xi}\bar{\Omega}(z)dz \tag{A2}$$

are given in two forms. The first form, which involves a series of Bessel functions, enables one to find the nature of the singularity. The other form gives the singularity explicity and is more convenient for computational purposes, as in the numerical integration of equations (38), (63), and (76).

Evaluation of the line integral (A2) can be obtained in the usual way by transforming the path of integration into a closed contour and applying the calculus of residues. The integrand possesses a simple pole at the origin and an infinite number of poles (roots of I_1) along the imaginary axis. The integrand also has a branch point at $z = 0$ due to K_1, so a closed circuit is chosen as indicated in sketch (r). Since for a number $\alpha > 0$



Sketch (r)

$$|\bar{\Omega}(Re^{i\theta})| < \frac{M}{R^{\alpha}}, \qquad -\pi \le \theta \le \pi \tag{A3}$$

The integral taken over the ABD and GHI of the circle c_1 of radius R can be shown to go to zero as $R \to \infty$. According to Cauchy's residue theorem, the evaluation of equation (A2) is then

[1]This influence function was derived and first investigated in some unpublished work of Max. A. Heaslet of Ames Aeronautical Laboratory.

$$\Omega(\xi) = \sum r_m(\xi) - Q(\xi) \tag{A4}$$

where

$$Q(\xi) = \frac{1}{2\pi i}\int_{DE} e^{z\xi}\bar{\Omega}(z)dz + \frac{1}{2\pi i}\int_{FG} e^{z\xi}\Omega(z)dz + \frac{1}{2\pi i}\lim_{\epsilon\to 0}\int_{-\pi}^{\pi} e^{\xi\epsilon e^{i\theta}}\Omega(\epsilon e^{i\theta})d(\epsilon e^{i\theta}) \tag{A5}$$

and r_m is the residue of $e^{z\xi}\bar{\Omega}(z)$ at $z = \pm i\lambda_m$, the quantity λ_m being the mth root of $J_1(\lambda) = 0$. We now consider the two terms on the right of equation (A4) separately.

Since

$$\frac{d}{dz}[zI_1(z)] = zI_o(z)$$

a particular member r_m under the summation sign in equation (A4) is given by

$$r_m = \frac{1}{\pi i\lambda_m}\left[\frac{e^{i\lambda_m(\xi+2)}K_1(i\lambda_m)}{I_o(i\lambda_m)} - \frac{e^{-i\lambda_m(\xi+2)}K_1(-i\lambda_m)}{I_o(-i\lambda_m)}\right] \tag{A6}$$

Use of relations

$$K(\pm i\lambda_m) = \pm\frac{\pi i}{2}Y_1(\lambda_m) \;;\qquad I_o(\pm i\lambda_m) = J_o(\lambda_m) \tag{A7}$$

then yields the result

$$\sum r_m = \sum_{m=1}^{\infty}\frac{Y_1(\lambda_m)}{\lambda_m J_o(\lambda_m)}\cos(\xi + 2)\lambda_m \tag{A8}$$

for the total contribution of the residues.

The last term of equation (A4), which represents the contribution resulting from integration over the path DEFG in sketch (r), may be written from equation (A5) in the form

$$Q(\xi) = \frac{1}{2\pi^2 i} \lim_{\epsilon \to 0} \left\{ \int_{\infty}^{\epsilon} \frac{e^{\xi t e^{i\pi}}}{t} \left[\frac{e^{2te^{i\pi}} K_1(te^{i\pi})}{I_1(te^{i\pi})} - \pi \right] dt + \right.$$

$$\left. \int_{\epsilon}^{\infty} \frac{e^{\xi t e^{-i\pi}}}{t} \left[\frac{e^{2te^{-i\pi}} K_1(te^{-i\pi})}{I_1(te^{-i\pi})} - \pi \right] dt + i \int_{\pi}^{\pi} e^{\xi \epsilon e^{i\theta}} \left[\frac{e^{2\epsilon e^{i\theta}} K_1(\epsilon e^{i\theta})}{I_1(\epsilon e^{i\theta})} - \pi \right] d\theta \right\} \tag{A9}$$

If the relation

$$\frac{K_1(te^{\pm i})}{I_1(te^{\pm i})} = \frac{K_1(t)}{I_1(t)} \pm i\pi \tag{A10}$$

is employed, equation (A9) then gives

$$Q(\xi) = \frac{1}{2\pi} \lim_{\epsilon \to 0} \left[2\pi - 2\int_{\epsilon}^{\infty} \frac{e^{-t(\xi+2)}}{t} dt + \frac{1}{\pi} \int_{\pi}^{-\pi} e^{(\xi+2)\epsilon e^{i\theta}} \frac{K_1(\epsilon e^{i\theta})}{I_1(\epsilon e^{i\theta})} d\theta \right] \tag{A11}$$

or

$$Q(\xi) = \frac{1}{\pi} \left[\pi + \frac{3}{4} - (\xi + 2)^2 + \ln 2 (\xi + 2) \right] \tag{A12}$$

Finally, using relation (A4), (A8), and (A12), we have the equation

$$\Omega(\xi) = \frac{1}{\pi} \left[\pi \sum_{m=1}^{\infty} \frac{Y_1(\lambda_m)}{\lambda_m J_0(\lambda_m)} \cos(\xi + 2)\lambda_m - \pi - \frac{3}{4} + (\xi + 2)^2 - \ln 2(\xi + 2) \right] \tag{A13}$$

for the inverse transform of the influence function.

The nature of the singularities in this expression may be found heuristically as follows. Except for the last term, which has a logarithmic infinity at $\xi = -2$, the only possible singularities in equation (A13) come from the series involving Bessel functions. Since the singularities due to this series are not affected by omitting the first N terms, we substitute the asymptotic expansions (ref. 6)

$$\left.\begin{aligned}
\lambda_m &\simeq \left(m+\frac{1}{4}\right)\pi - \frac{3}{8\pi\left(m+\frac{1}{4}\right)} + \frac{3}{128\pi^3\left(m+\frac{1}{4}\right)^3} + \ldots \\
J_0(\lambda_m) &\simeq \sqrt{\frac{2}{\pi\lambda_m}}\left[\cos\left(\lambda_m - \frac{\pi}{4}\right) + \frac{1}{8\lambda_m}\sin\left(\lambda_m - \frac{\pi}{4}\right) + \ldots\right] \\
Y_1(\lambda_m) &\simeq \sqrt{\frac{2}{\pi\lambda_m}}\left[-\cos\left(\lambda_m - \frac{\pi}{4}\right) + \frac{3}{8\lambda_m}\sin\left(\lambda_m - \frac{\pi}{4}\right) + \ldots\right]
\end{aligned}\right\} \tag{A14}$$

so that

$$\pi\sum_{m=N}^{\infty}\frac{Y_1(\lambda_m)}{\lambda_m J_0(\lambda_m)}\cos(\xi+2)\lambda_m = \pi\sum_{m=N}^{\infty}\frac{\cos(\xi+2)\lambda_m}{\lambda_m}\left[-1+O\left(\frac{1}{\lambda_m}\right)\right]$$

$$\simeq -\sum_{m=N}^{\infty}\frac{\cos\left[\left(m+\frac{1}{4}\right)(\xi+2)\pi\right]}{m+\frac{1}{4}} = \sum_{m=N}^{\infty}\frac{\sin\left(m+\frac{1}{4}\right)\pi\xi}{m+\frac{1}{4}} \tag{A15}$$

when N is sufficiently large. Now

$$\sum_{m=1}^{\infty}\frac{\sin\left(m+\frac{1}{4}\right)\pi\xi}{m+\frac{1}{4}} \simeq \frac{\pi}{2} - 4\sin\frac{\pi\xi}{4} + \frac{1}{2}\ln\frac{1+\sin\frac{\pi\xi}{4}}{1-\sin\frac{\pi\xi}{4}} \tag{A16}$$

which, near $\xi = \pm 2$ behaves like $\ln(2+\xi)/(2-\xi)$. One can see from equation (A13) that the function Ω has only a logarithmic singularity at $\xi = 2$.

EVALUATION OF $\Omega(\xi)$ BY ALTERNATE METHOD

The expression (A13) for the influence function Ω is unsuitable for numerical work, so that it is desirable to have an equation which is more convenient and which also isolates the singular part. We therefore seek an evaluation in the form

$$\Omega(\xi) = -\frac{1}{\pi}\ln\frac{2-\xi}{2} + T(\xi)\ , \qquad 0 \leq \xi < 2 \tag{A17}$$

where T is a convergent power series

$$T(\xi) = \sum_{j=0}^{\infty} a_j \xi^j \tag{A18}$$

If the function $\bar{\Omega}(z)$ in the line integral

$$\Omega(\xi) = \frac{1}{2\pi i} \int_{a-i\omega}^{a+i\omega} e^{z\xi}\bar{\Omega}(z)dz \tag{A19}$$

is expanded as an asymptotic series, the desired series expansion of $\Omega(\xi)$ can be obtained through term-by-term integration.

For large values of $|z|$, asymptotic expansions of K_1, I_1 are

$$\left.\begin{aligned} K_1(z) &\simeq \sqrt{\frac{\pi}{2z}}\, e^{-z} F\left(\frac{1}{z}\right); & |\arg z| &< 3\pi/2 \\ I_1(z) &\simeq \frac{e^z}{\sqrt{2\pi z}}\left[F\left(-\frac{1}{z}\right) - ie^{-2z}F\left(\frac{1}{z}\right)\right], & -\pi/2 < \arg z &< \pi/2 \end{aligned}\right\} \tag{A20}$$

where

$$\left.\begin{aligned} F\left(\frac{1}{z}\right) &= 1 + \sum_{m=1}^{\infty} \frac{b_m}{(2z)^m} \\ b_m = (1,m) &= \frac{(-1)^{m+1}(2m+1)[(2m)!]^2}{16^m(2m-1)(m!)^3} = \frac{(2m+1)(3-2m)}{4^m}\, b_{m-1} \end{aligned}\right\} \tag{A21}$$

so that for small ξ equation (A19) yields

$$\Omega(\xi) \simeq \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{e^{z\xi}}{z}\left[\frac{F(1/z)}{F(-1/z) - ie^{-2z}F(1/z)} - 1\right]dz \tag{A22}$$

The term $e^{-2z}F(1/z)$ in the denominator can be neglected if $\operatorname{Re}(z)$ (i.e., if a) is chosen sufficiently large and positive. Thus

$$\Omega(\xi) \approx \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{e^{\xi z}}{z}\left[\frac{F(1/z)}{F(-1/z)} - 1\right]dz \tag{A23}$$

Expanding the bracketed term as a series in 1/2z, one obtains

$$\Omega(\xi) \approx \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{z\xi}\sum_{m=1}^{\infty}\frac{c_m}{2^m}\frac{dz}{z^{m+1}}$$

or, after term-by-term integration

$$\Omega(\xi) \approx \sum_{m=1}^{\infty}\frac{c_m}{m!}\left(\frac{\xi}{2}\right)^m, \qquad 0 \leq \xi < 2 \tag{A24}$$

where the coefficients c_m may be determined in succession from the relations

$$\left.\begin{aligned}
&c_1 = 2b_1\\
&c_2 = c_1b_1\\
&c_3 = c_2b_1 - c_1b_2 + 2b_3\\
&\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\\
&c_m = c_{m-1}b_1 - c_{m-2}b_2 + \ .\ .\ . + (-1)^m c_1b_{m-1} + [1 + (-1)^{m+1}]b_m
\end{aligned}\right\} \tag{A25}$$

If use is now made of the expansion

$$\ln(2 - \xi) = \ln 2 - \sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\xi}{2}\right)^m \tag{A26}$$

it follows that the series T in equations (A17) and (A18) is given by

$$T(\xi) = \sum_{m=1}^{\infty}\frac{1}{m}\left[\frac{c_m}{(m-1)!} - \frac{1}{\pi}\right]\left(\frac{\xi}{2}\right)^m, \qquad 0 \leq \xi < 2 \tag{A27}$$

and that the influence function Ω can be expressed in the final form

$$\Omega(\xi) = -\frac{1}{\pi} \ln \frac{2-\xi}{2} + \sum_{m=1}^{\infty} \frac{1}{m} \left[\frac{c_m}{(m-1)!} - \frac{1}{\pi}\right]\left(\frac{\xi}{2}\right)^m, \qquad 0 \leq \xi < 2 \tag{A28}$$

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